A Deterministic Annealing Algorithm for a Combinatorial Optimization Problem by the use of Replicator Equations

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ABSTRACT

We have proposed an optimization method for a combinatorial optimization problem based on successive bifurcation characteristics of a replicator equation, and to obtain a good approximate solution of the optimization problem, a deterministic annealing algorithm has been applied. During the annealing process, bifurcations of equilibrium solutions occur and the bifurcation structure affects the performance of the deterministic annealing algorithm. In this paper, the bifurcation structure of the proposed dynamic system is analyzed in detail. It is shown that only pitchfork bifurcations occur in the annealing process, and the solution obtained by the annealing is the branch uniquely connected with the uniform solution. In many cases, the solution corresponds to a good approximate solution of the optimization problem.

1. INTRODUCTION

A combinatorial optimization problem is an optimization problem in which decision variables are discrete. Since the number of feasible solutions is finite, basically it is possible to obtain the optimal solution by computing values of performance index corresponding to all feasible solutions. But, since the number of feasible solutions increases exponentially with the size of the problem, such enumeration becomes impossible. Therefore, a variety of approximation methods have been proposed [1]. Among them, there is a method called an artificial neural network model. One of the models is the Hopfield model [2], in which a potential function is designed based on the performance index and the constraints of the problem, and then a dynamic system is constructed as a gradient system of the potential function. An approximate optimal solution of the optimization problem is obtained as a stable equilibrium solution of the dynamic system. In the Hopfield model, although the performance of the approximate solution is sensitive to the values of parameters in the potential function, there is no guide to determine these values. So a lot of trials to overcome the drawback have been done. One of them is the mean field annealing or the deterministic annealing [3, 4], which was originally derived as the mean field approximation of the simulated annealing. In this method, the parameter in the potential function is varied gradually to prevent the state of the system from being trapped in local minima and obtain an approximate optimal solution. On the other hand, we have proposed a method by the use of successive bifurcation characteristics of a class of dynamic systems [5]. The proposed method has been modeled by the use of replicator equations, and a deterministic annealing algorithm has been used to obtain an approximate optimal solution; The model has a uniform solution where a control parameter in the dynamic system is small. By increasing the value of the control parameter, equilibrium solutions corresponding to approximate solutions of an optimization problem bifurcate successively according to the value of the performance index. Therefore a good approximate solution can be obtained by gradually increasing the control parameter to the bifurcation point. In this paper, we analyze the bifurcation structure of the dynamic model in detail.

In this paper, the analysis is carried out with the quadratic assignment problem (QAP) [6] which is introduced in Sec. 2. In Sec. 3, the dynamic system which we have proposed is mentioned briefly. In Sec. 4, the bifurcation of the proposed model is analyzed, and in Sec. 5, based on the analysis, a deterministic annealing algorithm is proposed and applied to many instances of the QAP to verify the performance of the algorithm. Finally, we briefly summarize the results in Sec. 6.

2. THE QUADRATIC ASSIGNMENT PROBLEM

The Quadratic Assignment Problem (QAP) [6] is defined as follows:

$$\min_{p \in \Pi_{\mathcal{N}}} L(p), \tag{1}$$

$$L(p) = \sum_{i,j} a_{ij} b_{p(i)p(j)}, \tag{2}$$

where $A = (a_{ij})$ and $B = (b_{kl})$ are $N \times N$ matrices, $\Pi_{\mathcal{N}}$ is the set of all permutations of $\mathcal{N} = \{1, \cdots, N\}$ and p is an element of it. Using permutation matrices, the QAP can be stated also as follows:

$$\min_{X \in \Pi_{N \times N}} L(x_{ij}), \tag{3}$$

$$L(x_{ij}) = \mathrm{trace}(A^T X^T B X) = \sum_{i,i',j,j'} a_{jj'} b_{ii'} x_{ij} x_{i'j'}. \tag{4}$$

where $\Pi_{N \times N}$ is the set of all $N \times N$ permutation matrices and $X = (x_{ij})$ is an element of it.

The QAP is considered one of the hardest combinatorial optimization problems. For a QAP, the number of feasible solutions is $N!$ and it increases explosively with N. So it is practically impossible to compute values of the performance index for all these solutions. Therefore, heuristic algorithms are needed to obtain good approximate solutions in relatively short time from a practical point of view.

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3. EQUILIBRIUM SOLUTION AND ITS STABILITY OF THE PROPOSED MODEL [5]

In this section we refer to the model we have proposed briefly. The model is given as a following replicator equation:

$$\dot{u}_{ij} = f_{ij}(u_{i'j'}, \alpha_0, \alpha_1)u_{ij}, \tag{5a}$$

$$f_{ij} = (1 - u_{ij}^2) - \frac{\alpha_0}{2}\Big[\sum_{i' \neq i} u_{i'j}^2 + \sum_{j' \neq j} u_{ij'}^2\Big] - \frac{\alpha_1}{2}\sum_{i',j'}(a_{jj'}b_{ii'} + a_{j'j}b_{i'i})u_{i'j'}^2 \tag{5b}$$

$$(i, j = 1, \cdots, N),$$

where the variable u_{ij} $(i, j = 1, \cdots, N)$ expresses the (i, j) element of a $N \times N$ matrix, and parameters $\alpha_0 > 0$ and $0 \leq \alpha_1 \ll 1$. The first term of the right hand side of Eq. (5) leads each u_{ij}^2 to unity. The second term represents the effect of competition with other elements which have the same subscript i or j, and the parameter α_0 controls the strength of the competition. The third term represents the inhibition of the appearance of solutions with low performance. This system is derived as a gradient system of a potential function V:

$$V = V_0 + V_1, \tag{6}$$

$$V_0 = \frac{1}{4}\sum_{i,j}(1 - u_{ij}^2)^2 + \frac{\alpha_0}{8}\sum_{i,j}\Big[\sum_{i' \neq i} u_{i'j}^2 + \sum_{j' \neq j} u_{ij'}^2\Big]u_{ij}^2, \tag{7}$$

$$V_1 = \frac{\alpha_1}{4}L(u_{ij}^2). \tag{8}$$

Let $\bar{u}_{ij}$ $(i, j = 1, \cdots, N)$ denote an equilibrium solution of Eq. (5) and define a set Γ of subscripts (i, j) $(i, j = 1, \cdots, N)$ as follows:

$$\Gamma = \{(i, j) \mid \bar{u}_{ij} \neq 0,\ i, j = 1, \cdots, N\}. \tag{9}$$

Then, equilibrium solutions of Eq. (5) are classified as follows:

$$\begin{cases} \text{(A) uniform solution} & : \Gamma = \Gamma^0, \\ \text{(B) transition solution} & : \Gamma = \Gamma^t \neq \Gamma^0, \Gamma^p (\forall p \in \Pi_N), \\ \text{(C) feasible solution} & : \Gamma = \Gamma^p\ (p \in \Pi_N), \end{cases} \tag{10}$$

where

$$\Gamma^0 = \{(i, j) \mid i, j = 1, \cdots, N\}, \tag{11}$$

$$\Gamma^p = \{(i, j) \mid i = p(j); j = 1, \cdots, N\}\ (p \in \Pi_N). \tag{12}$$

The uniform solution is the equilibrium solution of which all the elements have nonzero values. On the other hand, each of the feasible solutions corresponds to a permutation matrix and thus corresponds to an approximate solution of the QAP.

Stability conditions of the solutions are summarized as follows; The sufficient condition for the uniform solution to be stable is given as follows:

$$0 < \alpha_0 < 1 - \frac{\alpha_1}{2}\max_{i,j}\sum_{i',j'}(a_{jj'}b_{ii'} + a_{j'j}b_{i'i}). \tag{13}$$

The condition (13) is also the necessary condition when $\alpha_1 = 0$ (i.e. $\alpha_0 < 1$). On the other hand, all equilibrium solutions except for the uniform solution are unstable if

$$0 < \alpha_0 < \frac{1}{N-1}\Big\{1 - \frac{\alpha_1}{2}\max_{i,j}\sum_{i',j'}(a_{jj'}b_{ii'} + a_{j'j}b_{i'i})\Big\}. \tag{14}$$

Therefore, in the region where parameters α_0, α_1 are sufficiently small, only the uniform solution is stable. The sufficient condition for all feasible solutions to be stable is given as follows:

$$\alpha_0 > \frac{1}{1 - \alpha_1(N-1)}. \tag{15}$$

This condition is also the necessary condition when $\alpha_1 = 0$ (i.e. $\alpha_0 > 1$). On the other hand, all equilibrium solutions except for the feasible solutions are unstable if

$$\alpha_0 > 2, \quad \alpha_1 < \frac{2}{\max_{i,j}\sum_{i',j'}(a_{jj'}b_{ii'} + a_{j'j}b_{i'i})}. \tag{16}$$

Therefore, only feasible solutions are stable when α_1 is small and α_0 is large. Moreover, the necessary condition for a feasible solution corresponding to a permutation $p \in \Pi_N$ to be stable is given by the following equation:

$$\alpha_0 > 1 + \frac{\alpha_1}{N-1}(L(p) - \bar{L}(p)), \tag{17}$$

$$L(p) = \sum_{j,j'} a_{jj'}b_{p(j)p(j')}, \tag{18}$$

$$\bar{L}(p) = \frac{1}{2N}\sum_{i,j,j'}(a_{jj'}b_{ip(j')} + a_{j'j}b_{p(j')i}), \tag{19}$$

where $L(p)$ is the value of the performance index corresponding to the feasible solution and $\bar{L}(p)$ corresponds to the mean value of the performance indices in a neighborhood of the feasible solution. It has been clarified numerically that the mean value $\bar{L}(p)$ is almost constant for any feasible solution. Therefore, the condition (17) indicates that the feasible solutions become stable successively according to the value of the performance index.

4. BIFURCATION OF THE PROPOSED MODEL

In this section, we analyze the bifurcation structure of the equilibrium solutions of the dynamic system (5) with the parameter α_0 as the control parameter.

First, we consider the local bifurcation structure of the dynamic system (5). Choose a set Γ, and let $\bar{u}_{ij}$ $(i, j = \cdots, N)$ be the corresponding equilibrium solution. The existence of the equilibrium solution is checked by the condition that Eq. (A1) has a positive solution. If the equilibrium solution exists, stability of the solution is checked by Eq. (A11). Since the matrix D^Γ in Eq. (A11) is symmetric, all the eigenvalues of the matrix D^Γ are real and the stability of the solution is checked by the sign of the eigenvalues. The stability of the solution changes at some point $\alpha_0 = \bar{\alpha}_0$ where the matrix D^Γ has a zero eigenvalue. At that point, the following equations hold:

$$\bar{f}_{i_k j_k} = 0 \ \text{ and } \ \bar{u}_{i_k j_k} = 0 \ \ (k = 1, \cdots, K). \tag{20}$$

Here, we assume that the zero eigenvalue is simple, i.e. $K = 1$. This assumption is valid unless the matrices A, B have

some specific structure (symmetry). When this assumption holds, the linearized system (A11) is rewritten as follows:

$$\begin{bmatrix} \delta\dot{\hat{u}}^{\Gamma} \\ \delta\dot{u}_{i_1 j_1} \end{bmatrix} = \begin{bmatrix} \hat{D}^{\Gamma} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \delta\hat{u}^{\Gamma} \\ \delta u_{i_1 j_1} \end{bmatrix}, \quad (21)$$

where the N^2-1-dimensional vector $\delta\hat{u}^{\Gamma}$ is given by removing the element $\delta u_{i_1 j_1}$ from the N^2-dimensional vector δu^{Γ} defined by Eq. (A12) and $\hat{D}^{\Gamma}$ is $(N^2-1)\times(N^2-1)$ matrix given by removing the row and the column corresponding to (i_1, j_1) from the matrix D^{Γ}. There exists a one-dimensional center manifold W^c tangent to a one-dimensional center subspace spanned by the eigenvector corresponding to the zero eigenvalue [7]. The center manifold W^c can be represented as a local graph:

$$W^c = \{(\delta\hat{u}^{\Gamma}, \delta u_{i_1 j_1}, \delta\alpha_0) | \delta\hat{u}^{\Gamma} = \boldsymbol{g}(\delta u_{i_1 j_1}, \delta\alpha_0)\}, \quad (22)$$

where $\delta\alpha_0 = \alpha_0 - \bar{\alpha}_0$ and $\boldsymbol{g}(\delta u_{i_1 j_1}, \delta\alpha_0)$ is a N^2-1-dimensional vector valued function of $\delta u_{i_1 j_1}$ and $\delta\alpha_0$ defined on some neighborhood of the bifurcation point $(\bar{u}_{ij}, \bar{\alpha}_0)$. The graph $\boldsymbol{g}(\delta u_{i_1 j_1}, \delta\alpha_0)$ is expanded in a Taylor series at the bifurcation point as follows:

$$\boldsymbol{g} = \delta u_{i_1 j_1}^2 \boldsymbol{w}_u^{\Gamma} + \delta\alpha_0 \boldsymbol{w}_{\alpha}^{\Gamma} + O(|\delta u_{i_1 j_1}|^3). \quad (23)$$

The coefficient vectors $\boldsymbol{w}_u^{\Gamma}, \boldsymbol{w}_{\alpha}^{\Gamma}$ are determined by the following equations:

$$\begin{cases} \hat{D}^{\Gamma}\boldsymbol{w}_u^{\Gamma} = \boldsymbol{h}_u^{\Gamma}, \\ \hat{D}^{\Gamma}\boldsymbol{w}_{\alpha}^{\Gamma} = \boldsymbol{h}_{\alpha}^{\Gamma}, \end{cases} \quad (24)$$

where the vectors $\boldsymbol{h}_u^{\Gamma}, \boldsymbol{h}_{\alpha}^{\Gamma}$ are given by Eqs. (A17),(A18). The one-dimensional reduced dynamic system on the center manifold W^c, which determines the local behavior of the dynamic system (5) near the bifurcation point, is given by

$$\delta\dot{u}_{i_1 j_1} = \mu_1(\delta\alpha_0 + \frac{\mu_2}{\mu_1}\delta u_{i_1 j_1}^2)\delta u_{i_1 j_1} + O(|\delta u_{i_1 j_1}|^4), \quad (25)$$

where constants μ_1, μ_2 are given by Eqs. (A23),(A24). The normal form (25) of the system shows that the bifurcation is the pitchfork bifurcation. There are two types of pitchfork bifurcations, i.e. the supercritical and the subcritical pitchfork bifurcations. By increasing the control parameter α_0, the equilibrium solution of the dynamic system (5) changes from the uniform solution to the feasible solution through the pitchfork bifurcations at some bifurcation points.

Next, we consider the global bifurcation structure of the dynamic system (5). Choose a set Γ. In the case where $\Gamma = \Gamma^t$ (the transition solution), by increasing α_0, the solution appears through a supercritical or a subcritical pitchfork bifurcation and disappears through a supercritical or a subcritical pitchfork bifurcation. In the case where $\Gamma = \Gamma^0$ (the uniform solution), the solution exists in the region where α_0 is small. By increasing α_0, the solution disappears through a supercritical pitchfork bifurcation. On the other hand in the case where $\Gamma = \Gamma^p$ (the feasible solution), the solution exists in the region where α_0 is large. By decreasing α_0, the solution disappears through a supercritical or a subcritical pitchfork bifurcation. When a solution disappears through a supercritical pitchfork bifurcation, there exists a solution which appears through the supercritical pitchfork bifurcation. These two solutions form a continuous branch. If a solution disappears through a subcritical pitchfork bifurcation when α_0 is increased, by slightly increasing α_0 the solution jumps and connects with another existing solution with smaller value of the potential function V. On the other hand, if a solution appears through a subcritical pitchfork bifurcation when α_0 is increased, by slightly decreasing α_0 the solution jumps and connects with another existing solution with smaller V value. The uniform solution connects with one feasible solution through supercritical and subcritical pitchfork bifurcations when α_0 is increased. On the other hand, feasible solutions connect with the uniform solution through supercritical and subcritical pitchfork bifurcations when α_0 is decreased. Figures 1 (a), (b) show two examples of the global bifurcation diagrams, where $N = 5$ and matrices A, B are randomly generated. In the figures, two types of branches are drawn; One is a branch which starts from the uniform solution and connects with one feasible solution when α_0 is increased. The others are branches which start from all feasible solutions and disappear through subcritical pitchfork bifurcations when α_0 is decreased. Particularly, the branch starting from the feasible solution corresponding to the optimal solution is drawn until it discontinuously connects with the uniform solution through a subcritical pitchfork bifurcation. In the case of Fig. 1(a), the uniform solution continuously connects with a feasible solution only through supercritical pitchfork bifurcations. On the other hand in the case of Fig. 1(b), the uniform solution discontinuously connects with a feasible solution through supercritical and subcritical pitchfork bifurcations. If there is a feasible solution with better performance than the feasible solution with which the uniform solution connects, the branch with which the better feasible solution connects crosses the branch with which the uniform solution connects when the parameter α_0 is decreases. In many cases, there are few branches crossing the branch with which the uniform solution connects. This indicates that the feasible solution with which the uniform solution connects is a good approximate solution, even if it is not optimal.

5. DETERMINISTIC ANNEALING ALGORITHM

To obtain a good approximate solution of a combinatorial optimization problem, there is a method called a deterministic annealing algorithm. For our model, this method is carried out as follows; First, the control parameter α_0 is set sufficiently small. Equation (5) is computed to obtain the uniform solution. Then, α_0 is slightly increased, and with the solution as the initial value, Eq. (5) is computed again to obtain the equilibrium solution. By repeating this procedure, a feasible solution is finally obtained. From the previous analysis of the bifurcation structure, it is considered that applying this method to our model, the feasible solution with which the uniform solution connects is obtained, and the solution is a good approximate solution. To accurately trace the branch connecting the uniform solution with a feasible solution, the increment of the parameter α_0 must be sufficiently small. But too small increment of α_0 increases the computational time. Therefore, to control the increment of the parameter α_0, we introduce the following parameter S:

$$S = -\frac{1}{N\log N}\sum_{i,j} p_{ij}\log p_{ij}, \quad p_{ij} = \frac{u_{ij}^2}{\sum_{j'} u_{ij'}^2}, \quad (26)$$

where p_{ij} is normalized so that the sum over j is equal to unity. The parameter S takes the maximum value 1 for the uniform solution and the minimum value 0 for feasible solutions. Figures 2 (a), (b) show the change of the parameter S for the branch connecting the uniform solution with a feasible solution when α_0 is increased. The cases (a)

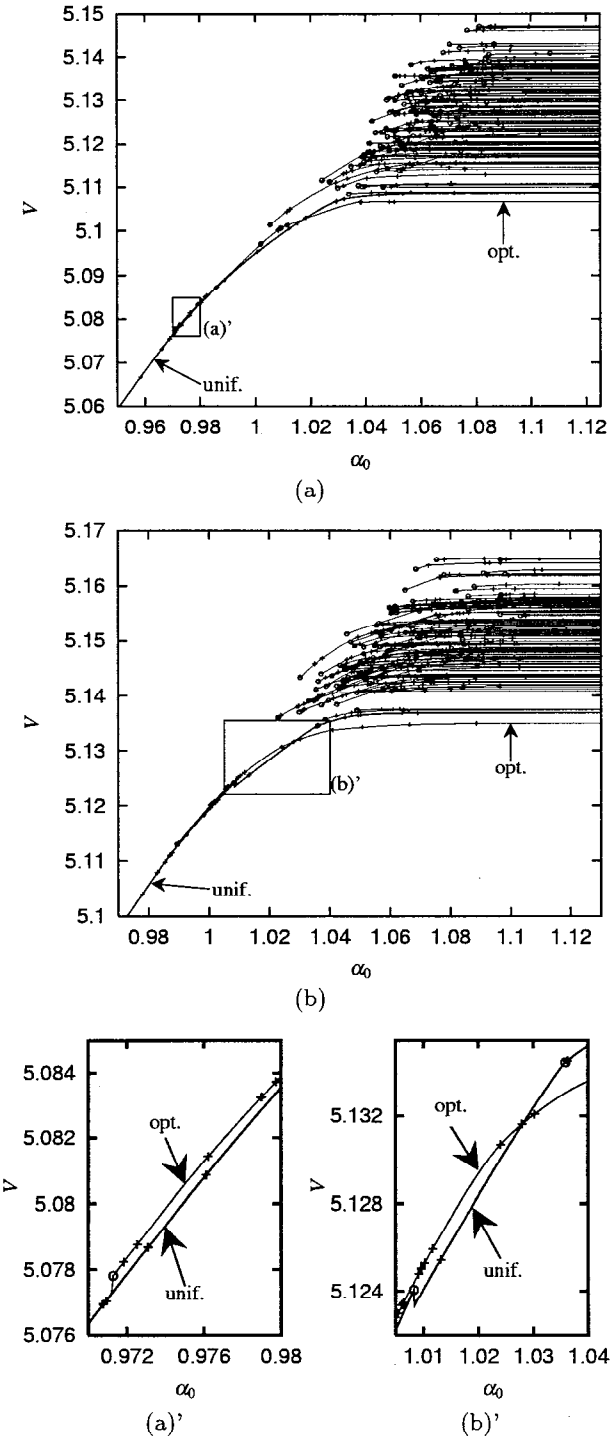

Fig. 1. The global bifurcation diagrams. Lines (—) indicate branches of equilibrium solutions. At the points indicated by (+) and (o), the supercritical and the subcritical pitchfork bifurcations occur respectively. "unif." represents the branch with which the uniform solution connects, and "opt." represents the branch with which the optimal feasible solution connects.

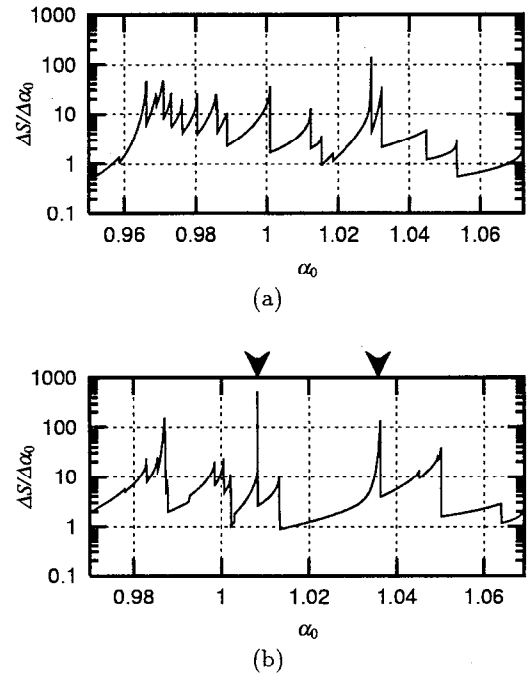

Fig. 2. The change of the parameter S with respect to the parameter α_0. Arrows in figure (b) indicate the points where the subcritical pitchfork bifurcations occur.

and (b) in Fig. 2 correspond to the cases (a) and (b) in Fig. 1 respectively. These figures show that the parameter S changes greatly at the point where a bifurcation occurs. Especially, the amount of the change is largest at the point where a subcritical pitchfork bifurcation occurs. Then, using the parameter S, the increment of the parameter α_0 is controlled, so that the amount of the change of S is kept constant;

$$\Delta\alpha_0 = \frac{\Delta S^{\mathrm{d}}}{\Delta S}\Delta\alpha_0^{\mathrm{old}}, \tag{27}$$

$$\Delta\alpha_0 = \alpha_0 - \alpha_0^{\mathrm{old}}, \quad \Delta S = |S - S^{\mathrm{old}}|, \tag{28}$$

where ΔS^{d} is the desired change of the parameter S and $\alpha_0^{\mathrm{old}}, S^{\mathrm{old}}$ mean the previous values of each parameter. By using the parameter S, the parameter α_0 is increased more slowly at points where the subcritical pitchfork bifurcations occur. And then, the annealing can be done efficiently. The results of the deterministic annealing algorithm for above examples are shown in Figs. 3 (a), (b). In each figure, the values of the performance index L of 100 trials of the deterministic annealing algorithm are plotted for various values of ΔS^{d} where the initial values are generated randomly. L_{opt} is the optimal value of L. In the case (a) where no subcritical pitchfork bifurcation occurs, the feasible solution with which the uniform solution connects is obtained for any value of ΔS^{d} and in any trial. On the other hand, in the case (b) where subcritical pitchfork bifurcations occur, another feasible solution than the above solution is obtained when ΔS^{d} is large. But by setting ΔS^{d} sufficiently small, the feasible solution with which the uniform solution connects is always obtained.

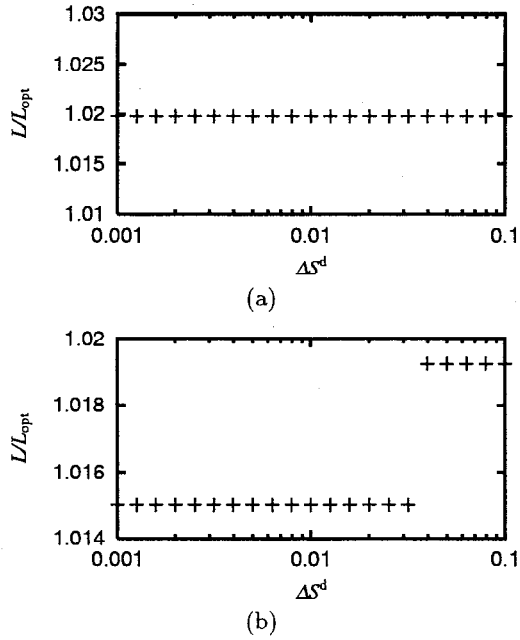

Fig. 3. The values of $L/L_{\rm opt}$.

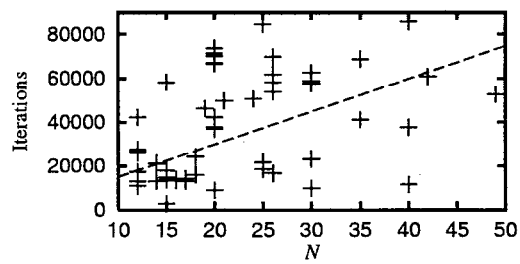

Fig. 4. The dependence of the number of steps on the problem size N. The dashed line(--) is the interpolating line.

Next, the deterministic annealing algorithm is applied to many problem instances in the QAPLIB [8], and some of the results are shown in Table 1. Table 1 shows that for many problem instances the difference between the value of the performance index obtained by the deterministic annealing algorithm and the optimal value is less than 1%, and for others the difference is within a few percent. Figure 4 represents the dependence of the number of steps on the problem size N. A linear interpolation of the data gives the relation, No. of Iterations $= 1586N$. This means that the number of steps doesn't depend on the problem size N very much. The computational time required by one step computation is proportional to about N^4. The total time required for computation is proportional to N^5. For instance, it is about 0.005 sec. (user time) for $N = 20$ on DEC Alpha Station 500/333, and therefore the feasible solution is obtained in about 2 min. for $N = 20$.

6. CONCLUSION

We have proposed an algorithm for a combinatorial optimization problem based on successive bifurcation characteristics of a replicator equation. In this paper, the bifurcation structure of the proposed dynamic system was analyzed in detail. It was shown that only pitchfork bifurcations occurr in this model. And, it was clarified that there is a feasible solution uniquely connected with the uniform solution, and the solution is a good approximate solution. We proposed a deterministic annealing algorithm which starts from the uniform solution and reaches the feasible solution connecting with it. Many results of application showed the high performance of this algorithm.

Table 1. The values of the performance index obtained by the deterministic annealing algorithm.

name	N	optimal solution† $L_{\rm opt}$	solution by the annealing L	$L/L_{\rm opt}$
bur26a	26	(5426670)	5439285	1.00232
Had20	20	6922	6970	1.00693
Nug20	20	2570	2588	1.007
Nug24	24	3488	3490	1.00057
Rou20	20	725522	730710	1.00715
Sko56	56	(34458)	34502	1.00128
Sko100a	100	(152002)	152502	1.00329
Tai50a	50	(4941410)	5051386	1.02226
Tai50b	50	(458821517)	459975270	1.00251
Tai80a	80	(13557864)	13733524	1.01296
Tai80b	80	(818415043)	821025553	1.00319
Tai100a	100	(21125314)	21557766	1.02047
Tai100b	100	(1185996137)	1193847431	1.00662
Tho30	30	(149936)	151256	1.0088
Tho40	40	(240516)	241192	1.00281
Tho150	150	(8133484)	8158137	1.00303
wil50	50	(48816)	48892	1.00156
wil100	100	(273038)	273294	1.00094

† (·):best known solution (optimality has not been proved)

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APPENDIX

Choose a set Γ, and let $\bar{u}_{ij}\,(i,j=1,\cdots,N)$ be the corresponding equilibrium solution. Then, since f_{ij} is linear with respect to u_{ij}^2, the equilibrium solution $\bar{u}_{ij}$ is obtained by solving the following linear algebraic equation [5]:

$$C^\Gamma \boldsymbol{z}^\Gamma = \boldsymbol{b}^\Gamma, \tag{A1}$$

where

$$\boldsymbol{z}^\Gamma = [\boldsymbol{x}^{\Gamma T}\boldsymbol{y}^{\Gamma T}]^T, \tag{A2}$$

$$\begin{aligned} \boldsymbol{x}^\Gamma &= [\cdots \bar{u}_{ij}^2 \cdots]^T \quad ((i,j)\in\Gamma),\\ \boldsymbol{y}^\Gamma &= [\cdots \bar{u}_{ij}^2 \cdots]^T \quad ((i,j)\notin\Gamma), \end{aligned} \tag{A3}$$

$$\boldsymbol{b}^\Gamma = [\overbrace{1\cdots 1}^{n(\Gamma)}\ \overbrace{0\cdots 0}^{N^2-n(\Gamma)}]^T, \tag{A4}$$

$$C^\Gamma = \begin{bmatrix} C_x^\Gamma & 0 \\ 0 & I_{N^2-n(\Gamma)} \end{bmatrix}, \tag{A5}$$

and C_x^Γ is a $n(\Gamma)\times n(\Gamma)$ matrix where $n(\Gamma)$ indicates the number of elements of the set Γ. The matrix C_x^Γ is given by removing columns and rows which correspond to $(i,j)\notin\Gamma$ from the following $N^2\times N^2$ matrix C:

$$C = C_0 + C_1, \tag{A6}$$

$$C_0 = \begin{bmatrix} C_0^{(1)} & C_0^{(2)} & \cdots & C_0^{(2)} \\ C_0^{(2)} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & C_0^{(2)} \\ C_0^{(2)} & \cdots & C_0^{(2)} & C_0^{(1)} \end{bmatrix}, \tag{A7}$$

$$C_0^{(1)} = \begin{bmatrix} 1 & \frac{\alpha_0}{2} & \cdots & \frac{\alpha_0}{2} \\ \frac{\alpha_0}{2} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{\alpha_0}{2} \\ \frac{\alpha_0}{2} & \cdots & \frac{\alpha_0}{2} & 1 \end{bmatrix}, \quad C_0^{(2)} = \frac{\alpha_0}{2} I_N, \tag{A8}$$

$$C_1 = \begin{bmatrix} C_1^{(11)} & \cdots & C_1^{(1N)} \\ \vdots & \ddots & \vdots \\ C_1^{(N1)} & \cdots & C_1^{(NN)} \end{bmatrix}, \tag{A9}$$

$$(C_1^{(jj')})_{ii'} = \frac{\alpha_1}{2}(a_{jj'}b_{ii'} + a_{j'j}b_{i'i}). \tag{A10}$$

By linearizing the dynamic system (5) with respect to $\delta u_{ij} = u_{ij} - \bar{u}_{ij}$, the following equation is obtained [5]:

$$\delta\dot{\boldsymbol{u}}^\Gamma = D^\Gamma \delta\boldsymbol{u}^\Gamma, \tag{A11}$$

where

$$\delta\boldsymbol{u}^\Gamma = [\delta\boldsymbol{u}_x^{\Gamma T}\delta\boldsymbol{u}_y^{\Gamma T}]^T, \tag{A12}$$

$$\begin{aligned} \delta\boldsymbol{u}_x^\Gamma &= [\cdots \delta u_{ij} \cdots]^T \quad ((i,j)\in\Gamma),\\ \delta\boldsymbol{u}_y^\Gamma &= [\cdots \delta u_{ij} \cdots]^T \quad ((i,j)\notin\Gamma), \end{aligned} \tag{A13}$$

$$D^\Gamma = \begin{bmatrix} D_x^\Gamma & 0 \\ 0 & D_y^\Gamma \end{bmatrix}, \tag{A14}$$

$$\begin{aligned} D_x^\Gamma &= -2P^\Gamma C_x^\Gamma P^\Gamma,\\ D_y^\Gamma &= \mathrm{diag}(\bar{f}_{ij}) \quad ((i,j)\notin\Gamma), \end{aligned} \tag{A15}$$

$$P^\Gamma = \mathrm{diag}(|\bar{u}_{ij}|)\ ((i,j)\in\Gamma). \tag{A16}$$

$\boldsymbol{h}_u^\Gamma, \boldsymbol{h}_\alpha^\Gamma$ in Eq. (24) are N^2-1-dimensional vectors given as follows:

$$\boldsymbol{h}_u^\Gamma = [\underbrace{\boldsymbol{h}_{ux}^{\Gamma T}}_{n(\Gamma)}\ \underbrace{\boldsymbol{h}_{uy}^{\Gamma T}}_{N^2-n(\Gamma)-1}]^T, \tag{A17}$$

$$\boldsymbol{h}_\alpha^\Gamma = [\overbrace{\boldsymbol{h}_{\alpha x}^{\Gamma T}}^{}\ \overbrace{\boldsymbol{h}_{\alpha y}^{\Gamma T}}^{}]^T, \tag{A18}$$

where

$$\begin{aligned} \boldsymbol{h}_{ux}^\Gamma &= [\cdots h_{u,ij}^\Gamma \cdots]^T \quad ((i,j)\in\Gamma),\\ \boldsymbol{h}_{uy}^\Gamma &= [0\cdots 0]^T, \end{aligned} \tag{A19}$$

$$\begin{aligned} \boldsymbol{h}_{\alpha x}^\Gamma &= [\cdots h_{\alpha,ij}^\Gamma \cdots]^T \quad ((i,j)\in\Gamma),\\ \boldsymbol{h}_{\alpha y}^\Gamma &= [0\cdots 0]^T, \end{aligned} \tag{A20}$$

$$\begin{aligned} h_{u,ij}^\Gamma &= -\frac{\bar{\alpha}_0}{2}(\delta_{ii_1}+\delta_{jj_1})\bar{u}_{ij}\\ &\quad -\frac{\alpha_1}{2}(a_{jj_1}b_{ii_1}+a_{j_1j}b_{i_1i})\bar{u}_{ij}\ ((i,j)\in\Gamma), \end{aligned} \tag{A21}$$

$$h_{\alpha,ij}^\Gamma = -\frac{1}{2}\Big[\sum_{\substack{i'\neq i\\(i',j)\in\Gamma}}\bar{u}_{i'j}^2 + \sum_{\substack{j'\neq j\\(i,j')\in\Gamma}}\bar{u}_{ij'}^2\Big]\bar{u}_{ij}\ ((i,j)\in\Gamma). \tag{A22}$$

The constants μ_1, μ_2 in Eq. (25) are given as follows:

$$\begin{aligned} \mu_1 &= -\bar{\alpha}_0\Big[\sum_{\substack{i'\neq i_1\\(i',j_1)\in\Gamma}}\bar{u}_{i'j_1}w_{\alpha,i'j_1} + \sum_{\substack{j'\neq j_1\\(i_1,j')\in\Gamma}}\bar{u}_{i_1j'}w_{\alpha,i_1j'}\Big]\\ &\quad -\alpha_1\sum_{\substack{i',j'\\(i',j')\in\Gamma}}(a_{j_1j'}b_{i_1i'}+a_{j'j_1}b_{i'i_1})\bar{u}_{i'j'}w_{\alpha,i'j'}\\ &\quad -\frac{1}{2}\Big[\sum_{\substack{i'\neq i_1\\(i',j_1)\in\Gamma}}\bar{u}_{i'j_1}^2 + \sum_{\substack{j'\neq j_1\\(i_1,j')\in\Gamma}}\bar{u}_{i_1j'}^2\Big], \end{aligned} \tag{A23}$$

$$\begin{aligned} \mu_2 &= -\bar{\alpha}_0\Big[\sum_{\substack{i'\neq i_1\\(i',j_1)\in\Gamma}}\bar{u}_{i'j_1}w_{u,i'j_1} + \sum_{\substack{j'\neq j_1\\(i_1,j')\in\Gamma}}\bar{u}_{i_1j'}w_{u,i_1j'}\Big]\\ &\quad -\alpha_1\sum_{\substack{i',j'\\(i',j')\in\Gamma}}(a_{j_1j'}b_{i_1i'}+a_{j'j_1}b_{i'i_1})\bar{u}_{i'j'}w_{u,i'j'}\\ &\quad -(1+\alpha_1 a_{j_1j_1}b_{i_1i_1}). \end{aligned} \tag{A24}$$