Attitude Control of a Spacecraft with Two Reaction Wheels

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Abstract

This paper deals with the attitude control of a rigid spacecraft with two re-action wheels. First, we derive a discontinuous state feedback law based on Lyapunov control. When the angular momentum of the system is zero, the derived controller makes the desired point the only stable equilibrium point of the system. Next, we investigate the behavior of the controlled system when the angular momentum is not zero but small.

Introduction

Two types of actuators, gas jet thrusters and reaction wheels, are commonly used for attitude control of spacecraft. These actuators are usually located to produce three independent torques about the principal axes of inertia of the spacecraft. Is the attitude control still possible when a fault of actuators occurs and a torque about one of the axes is disabled? This question is important from a practical point of view. In the case of reaction wheels, the system can be expressed as a nonholonomic system where the angular velocities of the wheels are control inputs [1],[2].

In this paper, attitude control of a spacecraft with two reaction wheels is discussed. First, we design a controller by extending the Lyapunov method [3] under the condition that the angular momentum of the system is zero. From a practical point of view, it is important to examine the influence of the residual angular momentum of the system on the performance of the controller. Next, the behavior of the controlled system is investigated in detail for the case where the angular momentum is not zero but small. In this case, the system converges to either a limit cycle or an equilibrium point which is not the desired point, but, in both cases, the error in attitude remains small.

Basic equation

We consider a spacecraft composed of three rigid bodies, the main body and two wheels, as shown in Fig. 1. The main body and the wheels are labeled as body 0, 1 and 2, respectively. We introduce a frame $\{a^{(-1)}\}$ fixed in an inertial space and a frame $\{a^{(i)}\}$ fixed in body i $(i = 0, 1, 2)$. The origin of $\{a^{(0)}\}$ is the total center of mass of the three bodies and can be assumed to be identical with the origin of $\{a^{(-1)}\}$. In this system, the angular momentum of the total system is conserved:

Figure 1: Schematic model of a spacecraft with two reaction wheels

$$J_t \omega^{(0,-1)} + j_1 \dot{\theta}_1 z_{w1} + j_2 \dot{\theta}_2 z_{w2} = A^{(0,-1)} H_0 \ , \tag{1}$$

where

J_t inertia of the total system about the origin of $\{a^{(0)}\}$ expressed in the frame $\{a^{(0)}\}$, $J_t = \mathrm{diag}\{J_{t1}, J_{t2}, J_{t3}\}$

H_0 residual angular momentum of the total system about the origin of $\{a^{(0)}\}$ expressed in the frame $\{a^{(-1)}\}$

z_{wi} rotation axis of wheel i expressed in the frame $\{a^{(0)}\}$ $(i = 1, 2)$ $z_{w1} = [1, 0, 0]^T$, $z_{w2} = [0, 1, 0]^T$

j_i inertia of wheel i about the rotation axis $(i = 1, 2)$

θ_i angle of rotation of wheel i $(i = 1, 2)$

$\omega^{(0,-1)}$:   angular velocity of $\{a^{(0)}\}$ with respect to $\{a^{(-1)}\}$ expressed in the frame $\{a^{(0)}\}$

$A^{(0,-1)}$:   a coordinate transform matrix from $\{a^{(-1)}\}$ to $\{a^{(0)}\}$

The equation for the rotation angles of the wheels is obtained as follows:

$$P \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} - \begin{bmatrix} j_1 z_{w1}^T J_t^{-1} \\ j_2 z_{w2}^T J_t^{-1} \end{bmatrix} \tilde{\omega}^{(0,-1)} A^{(0,-1)} H_0 \ , \tag{2}$$

where τ_i is the torque that drives wheel i $(i = 1, 2)$, $P = \mathrm{diag}\{j_1(1-j_1/J_{t1}), j_2(1-j_2/J_{t2})\}$, and $\tilde{p}^T$ is defined as $\tilde{p}^T = p\times$ for $p \in R^3$.

In this paper, the Euler parameters are used for expressing the attitude of the main body of the spacecraft: $e_0 = \cos(\phi/2)$, $e = [e_1, e_2, e_3]^T = a\sin(\phi/2)$, where $e_0^2 + e_1^2 + e_2^2 + e_3^2 = 1$. The derivatives of the Euler parameters become

$$\dot{e} = (1/2) \cdot (e_0 I - \tilde{e}) \omega^{(0,-1)} \equiv (1/2) \cdot Q \omega^{(0,-1)} \ . \tag{3}$$

When the variables $\dot{\theta}_1$ and $\dot{\theta}_2$ are transformed to the variables u_1 and u_2 as $u_i = -(j_i/J_{ti}) \cdot \dot{\theta}_i$ $(i = 1, 2)$, Eq.(1) can be rewritten in the form as

$$\omega^{(0,-1)} = [u_1, u_2, 0]^T + J_t^{-1} A^{(0,-1)} H_0 \ . \tag{4}$$

We make the following assumptions: The residual angular momentum H_0 is zero, u_1 and u_2 are input variables of the system, and the Euler parameters e
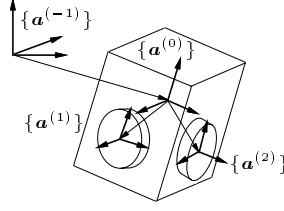
are the controlled variables. Based on Eqs.(3) and (4), the basic equation for controller design becomes

$$\dot{e} = \frac{1}{2} \begin{bmatrix} e_0 & -e_3 \\ e_3 & e_0 \\ -e_2 & e_1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \equiv B u \ . \tag{5}$$

The controller design

We have proposed a method to design a controller for a class of nonholonomic systems based on Lyapunov control [3]. A Lyapunov function is introduced as $V(e) = (1/2) \cdot (e_1^2 + e_2^2 + e_3^2)$, and the input vector is constructed as follows:

$$u = -(\alpha I - (\beta e_3/g^2) \cdot J) B^T \nabla V \ , \tag{6}$$

where α and β are positive constants, $\nabla = [\partial/\partial e_1, \partial/\partial e_2, \partial/\partial e_3]^T$,

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \ , \quad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \ , \quad g = |B^T \nabla V| = \frac{1}{2} e_0 \sqrt{(e_1^2 + e_2^2)} \ . \tag{7}$$

When Eq.(6) is used as input, the basic equation, Eq.(5), becomes

$$\dot{e} = -B (\alpha I - (\beta e_3/g^2) \cdot J) B^T \nabla V \ . \tag{8}$$

The analysis that was conducted in a similar way as in [3] shows that the system described by Eq.(8) converges to the origin exponentially. In the following, this result is summarized briefly.

With Eq.(8), the derivative of $V(t)$ is computed as $\dot{V} = -\alpha |B^T \nabla V|^2 \leq 0$. Therefore, the controlled system converges to the line, $B^T \nabla V = 0$, which coincides with the e_3 axis, $e = (0, 0, e_3)$, from Eq.(7). Since Eq.(8) is discontinuous on the axis, we modify Eq.(8) as follows:

$$\dot{e} = -B (\alpha I - (\beta e_3/g^2) \cdot \tanh(g^2/\epsilon) J) B^T \nabla V \ . \tag{9}$$

When ϵ is a small positive constant, by linearizing Eq.(9) near the e_3 axis, the stability of the system on the e_3 axis is revealed as follows: $|e_3| < \sqrt{\alpha \epsilon/\beta} \Longleftrightarrow$ stable focus, $|e_3| > \sqrt{\alpha \epsilon/\beta} \Longleftrightarrow$ unstable focus. As ϵ approaches zero, Eq.(9) approaches Eq.(8) and the origin becomes the only stable equilibrium point of the system. In the neighborhood of the origin, from Eq.(8), the approximate solutions of the variables e_3 and g can be obtained as $e_3 = C_0 e^{-\beta t}$, $g = \sqrt{C_0' e^{-\frac{\alpha}{2} t} + \beta C_0^2 e^{-2\beta t}/(\alpha - 4\beta)}$ (when $\alpha \neq 4\beta$) or $\sqrt{(C_0' + \beta C_0^2 t/2) e^{-2\beta t}}$ (when $\alpha = 4\beta$), where C_0 and C_0' are constants. From these solutions, the system converges to the origin exponentially. By taking account of the oscillation and the magnitude of the inputs, it is recommended to set the parameters α and β so that $\beta > \alpha/2$.

Behavior of the controlled system

In this section, the behavior of the controlled system is investigated for the case where the residual angular momentum, $H_0 = [h_1, h_2, h_3]^T$, is not zero but small. From Eqs.(3) and (4), the basic equation of the system with the input of Eq.(6) becomes

$$\dot{e} = -B(\alpha I - (\beta e_3/g^2) \cdot J)B^T \nabla V + (1/2) \cdot QJ_t^{-1}A^{(0,-1)}H_0 \ . \qquad (10)$$

We investigate the behavior of the system described by Eq.(10) near the origin ($|e_1|, |e_2|, |e_3| \ll 1$, $e_0 \approx 1$). Near the origin, the second term in the right-hand side of Eq.(10) can be approximated as $(1/2) \cdot QJ_t^{-1}A^{(0,-1)}H_0 = (1/2) \cdot J_t^{-1}H_0 \equiv [f_1, f_2, f_3]$, where f_1, f_2 and f_3 are the constants corresponding to h_1, h_2 and h_3. Then, the basic equation (10) becomes approximately

$$\frac{d}{dt}\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = -\frac{\alpha}{4}\begin{bmatrix} e_1 - e_3 e_2 \\ e_3 e_1 + e_2 \\ 0 \end{bmatrix} - \frac{\beta}{4}\frac{e_3}{g^2}\begin{bmatrix} -e_2 - e_3 e_1 \\ -e_3 e_2 + e_1 \\ e_1^2 + e_2^2 \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} \ . \qquad (11)$$

From Eq.(11), the approximate solution of the variable e_3 is obtained as $e_3 = Ce^{-\beta t} + f_3/\beta$, where C is a constant. On the other hand, by neglecting the small terms and using the solution of e_3, the derivative of the variable g can be calculated as follows:

$$\dot{g} = -\alpha g/4 + \beta(Ce^{-\beta t} + f_3/\beta)^2/(4g) + (e_1 f_1 + e_2 f_2)/(4g) \ . \qquad (12)$$

First, we show that the variables e stay in the small region near the origin. We introduce a small quantity f_0 such that $|f_1|, |f_2| < f_0$, a time T such that $|C|e^{-\beta t} < f_0$ for $\forall t > T$, and define a small quantity F_0 as $F_0 = 2f_0/\alpha + \sqrt{(2f_0/\alpha)^2 + \beta(f_0 + |f_3|/\beta)^2/\alpha}$. From Eq.(12), the relation that $\dot{g} < 0$ holds for $\forall t > T$ in the region where $g \geq F_0$. Therefore, with the passage of time, the variable g goes into the region where $g < F_0$. Since $g = \sqrt{e_1^2 + e_2^2}/2$, the system converges to the region where $\sqrt{e_1^2 + e_2^2} < 2F_0$ and $e_3 = f_3/\beta$.

Next, we proceed to analyze the behavior of the system near the origin. When the main body is in a controlled attitude, there are the following two types of behavior according to the conditions of the residual angular momentum H_0.

Case A: (The case where $|f_3| \leq O(f_1^2 + f_2^2)$.) When $f_3 = 0, f_1^2 + f_2^2 \neq 0$, the point expressed by $e_c = [4f_1/\alpha, 4f_2/\alpha, 0]^T$ is an equilibrium point of the system in Eq.(11). It can be shown that the equilibrium point is stable by linearizing Eq.(11) near the point. When f_3 is small but not zero, from Eq.(11), an equilibrium point $e_c = [e_{1c}, e_{2c}, f_3/\beta]$ satisfies approximately the following equation:

$$\begin{bmatrix} e_{1c} \\ e_{2c} \end{bmatrix} = \frac{-1}{(-\frac{\alpha}{4} + \frac{f_3^2}{4g_c^2\beta})^2 + (\frac{f_3}{4g_c^2})^2}\begin{bmatrix} -\frac{\alpha}{4} + \frac{f_3^2}{4g_c^2\beta} & -\frac{f_3}{4g_c^2} \\ \frac{f_3}{4g_c^2} & -\frac{\alpha}{4} + \frac{f_3^2}{4g_c^2\beta} \end{bmatrix}\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \ , \qquad (13)$$

where $g_c = \sqrt{e_{1c}^2 + e_{2c}^2}/2$. Substituting Eq.(13) into $g_c = \sqrt{e_{1c}^2 + e_{2c}^2}/2$, we obtain

$$\alpha^2 g_c^4 - 2(\alpha f_3^2/\beta + 2(f_1^2 + f_2^2))g_c^2 + f_3^4/\beta^2 + f_3^2 = 0 \ . \qquad (14)$$

There exist solutions of e_{1c} and e_{2c} which satisfy Eq.(13) only if there exists a solution of g_c which satisfies Eq.(14) and $g_c > 0$. From Eq.(14), the solution of g_c exists if and only if $\alpha^2 f_3^2 \le 4(f_1^2 + f_2^2)^2/(1 - 4(f_1^2 + f_2^2)/(\alpha\beta)) \approx 4(f_1^2 + f_2^2)^2$. As a result, when this inequality is satisfied, the variable g_c can be calculated from Eq.(14) as $g_c = g_{c1}$, or g_{c2} (assume without loss of generality that $g_{c2} \le g_{c1}$). The equilibrium point e_c can be calculated from Eq.(13) using the solution of g_{c1} or g_{c2}. By linearizing Eq.(11) near the equilibrium point, it is revealed that the equilibrium point is a stable node when $g_c = g_{c1}$ and a saddle point when $g_c = g_{c2}$. At the stable node, the main body is in an attitude that diverges by a small amount from the desired attitude, and each of the wheels rotates at a constant angular velocity.

Case B: (The case where $f_3 \ne 0$.) When $f_3 \ne 0$ and $f_1 = f_2 = 0$, the approximate solution of the variable g can be solved analytically from Eq.(12), and we can obtain that $e_3 \to e_{3c}$ and $g \to g_{c\ell}$ as $t \to \infty$, where $e_{3c} = f_3/\beta$, $g_{c\ell} = \sqrt{f_3^2/(\alpha\beta)}$. When $e_3 = e_{3c}$ and $g = g_{c\ell}$, from Eq.(11), we obtain

$$e_1 = \frac{2f_3}{\sqrt{\alpha\beta}} \sin(\frac{\alpha\beta}{4f_3}t + \psi) \ , \ e_2 = \frac{2f_3}{\sqrt{\alpha\beta}} \cos(\frac{\alpha\beta}{4f_3}t + \psi) \ , \qquad (15)$$

where ψ is a constant. Therefore, the system converges to the limit cycle which is expressed by Eq.(15) and $e_3 = f_3/\beta$. Next, it is shown that the limit cycle can exist and be stable even if $f_1^2 + f_2^2 \ne 0$. When $e_3 = e_{3c}$, we express the variables e_1 and e_2 as

$$e_1 = (\frac{2f_3}{\sqrt{\alpha\beta}}+r(t)) \sin(\frac{\alpha\beta}{4f_3}t+\psi(t)) \ , e_2 = (\frac{2f_3}{\sqrt{\alpha\beta}}+r(t)) \cos(\frac{\alpha\beta}{4f_3}t+\psi(t)) \ , \ (16)$$

and investigate the behavior of the variables $r(t)$ and $\psi(t)$. If $|r(t)| \ll |f_3|$, the derivatives of the variables $r(t)$ and $\psi(t)$ can be approximated as follows:

$$\dot{r}(t) = f_1 \sin(\frac{\alpha\beta}{4f_3}t + \psi(t)) + f_2 \cos(\frac{\alpha\beta}{4f_3}t + \psi(t)) - \frac{\alpha}{2}r(t) \ , \qquad (17)$$

$$\dot{\psi}(t) = \frac{\sqrt{\alpha\beta}}{2f_3}\Big\{ f_1 \cos(\frac{\alpha\beta}{4f_3}t + \psi(t)) - f_2 \sin(\frac{\alpha\beta}{4f_3}t + \psi(t)) \Big\} - \frac{(\alpha\beta)^{\frac{3}{2}}}{4f_3^2}r(t) \ . \ (18)$$

From Eqs.(17) and (18), it can be shown that $|r(t)| \ll |f_3|$ for $\forall t > 0$ and, consequently, the limit cycle is stable. When the system is in the limit cycle, each wheel rotates in one direction and the other periodically. The main body does not rotate around the vector expressed as $[0, 0, 1]^T$ in the frame $\{a^{(-1)}\}$, though the angular momentum h_3 exists.

Lastly, it should be noted that, from the above analysis, there may exist both of the two types of behavior of the system, an equilibrium point and a limit cycle, in the region where $\alpha^2 f_3^2 \le 4(f_1^2 + f_2^2)^2$. Moreover, it can be shown by analysis that $g_{c\ell} < g_{c2} \le g_{c1}$.

Numerical simulations

We executed numerical simulations based on the basic equation for control, Eq.(10). The initial values of the system and the residual angular momentum H_0 are set as shown in Table. 1. The parameters of the spacecraft and the controller are set as $J_t = \text{diag}\{500, 500, 500\}$ [kg · m^2] , $j_1 = j_2 = 10$ [kg · m^2] , $\alpha = 0.04$, $\beta = 0.02$. Also, numerical simulations based on the equation of motion (2) were conducted in order to check the effect of the dynamic characteristics of controller. In Eq.(2), we use the input torques defined as $[\tau_1, \tau_2]^T = -K_t P([\dot{\theta}_1, \dot{\theta}_2]^T - [\dot{\theta}_{1d}, \dot{\theta}_{2d}]^T)$, where $\dot{\theta}_{id}(e) = -(J_{ti}/j_i) \cdot u_i(e)$ ($i = 1, 2$) and $u(e)$ is given by Eq.(6). The feedback gain K_t is chosen as $K_t = 10.0[1/s]$. Each torque τ_i is assumed to be saturated when its absolute value reaches 1.0[N·m].

Table 1: Simulation cases

	Initial Value of e^T	H_0^T[kg · m^2/s]
Case 1	(0.3 , 0.3 , 0.3)	(0.1 , 0.1 , 0.0001)
Case 2	(0.3 , 0.3 , 0.3)	(0.0 , 0.0 , 0.1)
Case 3	(0.3 , 0.3 , 0.3)	(0.1 , 0.1 , 0.001)
Case 4	(0.05 , 0.0 , 0.1)	(0.1 , 0.1 , 0.001)

Case A: In Case 1, since $\alpha^2 f_3^2 < 4(f_1^2 + f_2^2)^2$, a stable equilibrium point exists and the system converges to the point. Figure 2 shows the behavior of the system based on the basic equation. At the equilibrium point, $(e_1, e_2, e_3) = (0.0102, 0.00974, 2.49 \times 10^{-6})$ and $g = 0.00707$. From the theoretical analysis, we obtain $e_{3c} = 5.0 \times 10^{-6}$ and $g_{c1} = 0.00706$. On the other hand, the system based on the equation of motion also converges to the same equilibrium point as in Fig. 2.

Case B: In Case 2, since $f_3 \neq 0$ and $f_1 = f_2 = 0$, the system converges to a limit cycle. Figure 3 shows the behavior of the system based on the basic equation. On the limit cycle, $e_3 = 0.005$, $g = 0.0035$. This result coincides with the theoretical analysis very well. The theoretically calculated wheel torques required during the limit cycle are about 28.3[N·m]. On the other hand, the system based on the equation of motion converges to a limit cycle where $e_3 = 0.005$, $g = 0.0105$. Because of the saturation of the torques and the time lag between the angular velocities of the wheels $\dot{\theta}_i$ and the reference velocities $\dot{\theta}_{id}$, the radius of the limit cycle becomes larger.

Lastly, we executed numerical simulations based on the basic equation for Case 3 and 4, where $\alpha^2 f_3^2 < 4(f_1^2 + f_2^2)^2$. Figures 4 and 5 show that the system converges to an equilibrium point or a limit cycle according to the initial values of the variables e. At the equilibrium point in Fig. 4, $(e_1, e_2, e_3) = (0.0118, 0.00683, 2.49 \times 10^{-5})$. On the limit cycle in Fig. 5, $e_3 = 5.0 \times 10^{-5}$, $g = 3.5 \times 10^{-5}$. The numerical simulations based on the equation of motion were also carried out. They show that the system converges to the equilibrium point in both Case 3 and 4. It may suggest that the limit cycle is not possible to realize because of the saturation and the time lag of controller.
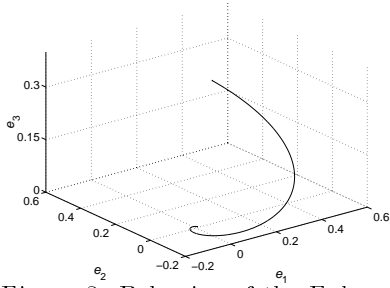
Figure 2: Behavior of the Euler parameters based on the basic equation (Case 1)
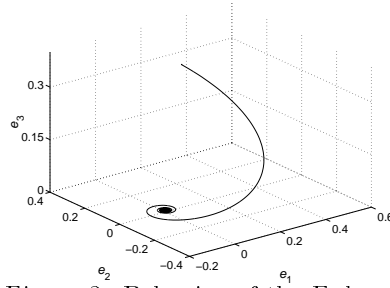

Figure 3: Behavior of the Euler parameters based on the basic equation (Case 2)
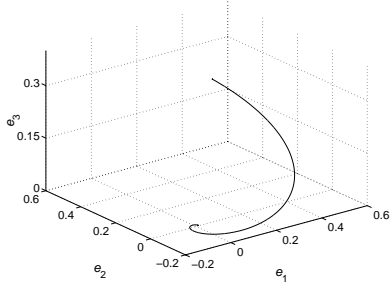

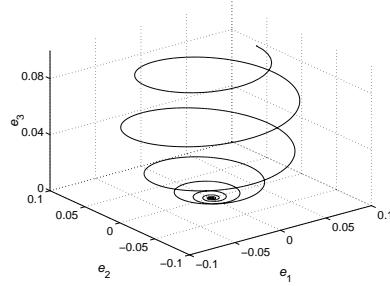Figure 4: Behavior of the Euler parameters based on the basic equation (Case 3)

Figure 5: Behavior of the Euler parameters based on the basic equation (Case 4)

Conclusions

In this paper, we discussed the attitude control of a spacecraft with two reaction wheels. A discontinuous state feedback controller which makes the attitude converge to the desired one when the angular momentum of the spacecraft is zero was designed. When the angular momentum is not zero but small, there exist two types of behavior to the controlled attitude, an equilibrium point and a limit cycle. This result was verified by analysis and numerical simulations.

References

[1] Krishnan, H., McClamroch, N.H., and Reyhanoglu, M., Attitude Stabilization of a Rigid Spacecraft Using Two Momentum Wheel Actuators, *Journal of Guidance, Control, and Dynamics*, vol.18, no.2, pp.256-263, 1995

[2] Leonard, N.E., and Krishnaprasad, P.S., Motion Control of Drift-Free, Left-Invariant Systems on Lie Groups, *IEEE Transactions on Automatic Control*, vol.40, no.9, pp.1539-1554, 1995

[3] Tsuchiya, K., Urakubo, T., and Tsujita, K., Motion Control of a Nonholonomic System Based on the Lyapunov Control Method, to appear in *Journal of Guidance, Control, and Dynamics*, vol.25, no.2, 2002