

Synthesis of Hybrid Systems with Limit Cycles Satisfying Piecewise Smooth Constraint Equations

Masakazu ADACHI^{†a)}, Toshimitsu USHIO^{†b)}, *Members,*
and Shigeru YAMAMOTO^{†c)}, *Nonmember*

SUMMARY In this paper, we propose a synthesis method of hybrid systems with specified limit cycles. Several methods which synthesize a nonlinear system with prescribed limit cycles have been proposed. In these methods, the limit cycle is given by an algebraic equation, which will be called constraint equations, and its stability is guaranteed by a Lyapunov function derived from the constraint equation. In general, limit cycles of hybrid systems are nonsmooth due to the discontinuous vector fields. So the limit cycles are given by piecewise smooth constraint equations, we employ the piecewise smooth Lyapunov functions to construct desired nonsmooth limit cycles and guarantee their stability.

key words: *limit cycles, hybrid systems*

1. Introduction

Limit cycles are one of the most important phenomena in nonlinear dynamical systems, and applied in many engineering fields. While stability analysis of limit cycles is a fundamental problem and many theories such as Lyapunov function methods have been proposed, the inverse problem of synthesizing a nonlinear system which has a stable and prescribed limit cycle is also important. Several methods for the inverse problem have been proposed [1]–[4].

On the other hand, dynamical systems whose states consist of both continuous and discrete variables are called hybrid dynamical systems [5]. Behaviors of their discrete states are piecewise-constant, and their continuous states evolve according to differential equations corresponding to the current discrete states. Behaviors of the continuous states are inherently nonsmooth because of change of the discrete states. Many physical and mechanical systems can be naturally described by hybrid systems. In hybrid systems, stability analysis is very difficult since continuous Lyapunov functions are no more useful. Recently, many approaches to stability analysis based on discontinuous Lyapunov functions or multiple ones have been developed [6]–[8].

Since several hybrid systems do not have a constant steady state but a periodic one, studies of limit cycles in hybrid systems are more important. For example,

the existence and stability of limit cycles in a switched server system [9], [10], and global asymptotical stability of limit cycles in relay feedback systems using extended Poincaré maps [11] have been reported. In more general cases, discrete-time model is derived by focusing on points where solutions hit switching surfaces, and it is possible to check the exponential convergence of limit cycle by using discrete-time Lyapunov functions [12]. However, there are little studies on how to construct a hybrid system which has a stable nonsmooth limit cycle. From an engineering viewpoint, such a limit cycle is applicable: for example, it can approximate walking patterns of humanoid robots [13], [14].

This paper proposes a synthesis method of hybrid systems with nonsmooth limit cycles. In the proposed method, a given limit cycle is split into some ellipsoidal curves, we calculate a piecewise smooth constraint function $V(x)$ derived from piecewise quadratic Lyapunov functions such that $V(x)$ is constant on each curve, and we obtain a desired hybrid system. The proposed method is an extension of Green's method [4].

This paper is organized as follows. In Section 2, we revisit and reformulate some useful techniques reported in [4] and show illustrative examples. In Section 3, hybrid systems derived from a piecewise smooth constraint equation are presented and we discuss their properties. An example illustrates the results.

2. Systems with Prescribed Limit Cycles

In this section, we present several concepts that will be used throughout this paper. First, we consider the following continuous differential equations:

$$\begin{aligned} \dot{x} &= f(x) + g(x), \\ f: \mathcal{R}^n &\rightarrow \mathcal{R}^n, g: \mathcal{R}^n \rightarrow \mathcal{R}^n. \end{aligned} \quad (1)$$

Green [4] shows sufficient conditions such that solutions $x(t)$ of (1) satisfy a given constraint $V(x) = 0$ as $t \rightarrow \infty$.

Theorem 1 ([4]): If there exists a continuously differentiable function $V: \Omega \rightarrow \mathcal{R}^m$ where Ω is a subset of \mathcal{R}^n , ($n > m$) such that

$$(i) \quad \frac{\partial V(x)}{\partial x} f(x) = 0, \quad \forall x \in \Omega.$$

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[†]The authors are with the Graduate School of Osaka University, Machikaneyama 1-3, Toyonaka, Osaka 560-8531

a) E-mail: adachi@hopf.sys.es.osaka-u.ac.jp

b) E-mail: ushio@sys.es.osaka-u.ac.jp

c) E-mail: yamamoto@sys.es.osaka-u.ac.jp

(ii) For each μ th component of V , $1 \leq \mu \leq m$,

$$\frac{\partial V_\mu(x)}{\partial x} g(x) V_\mu(x) < 0, \\ \forall x \in \Omega \text{ such that } V_\mu(x(t)) \neq 0,$$

then any solution $x(t) \in \Omega$ of (1) except equilibrium solutions satisfies $\lim_{t \rightarrow \infty} V(x(t)) = 0$. \square

From Theorem 1, if $V^{-1}(0)$ forms a closed curve and contains no equilibrium points, a solution of (1) starting from any initial point converges to the hypersurface $V(x) = 0$, and after the convergence the solution forms a closed curve on this hypersurface. As a special case, we consider that $m = 1$, and a system is described by the following form:

$$\dot{x} = Ax + a + V(x)(Bx + b), \quad (2)$$

where $x \in \mathcal{R}^n$, $A, B \in \mathcal{R}^{n \times n}$, $a, b \in \mathcal{R}^n$, and $V : \mathcal{R}^n \rightarrow \mathcal{R}$. In order to simplify the description of this system, we augment the state vector and rewrite (2) as

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + V(x)\tilde{B}\tilde{x}, \quad (3)$$

where $\tilde{x} = \begin{bmatrix} x \\ 1 \end{bmatrix}$, $\tilde{A} = \begin{bmatrix} A & a \\ 0 & 0 \end{bmatrix}$, and $\tilde{B} = \begin{bmatrix} B & b \\ 0 & 0 \end{bmatrix}$. A constraint function $V(x)$ is given by a Lyapunov function

$$V(x) = x^T P x + 2p^T x + \pi = \tilde{x}^T \tilde{P} \tilde{x}, \quad (4)$$

where $P \in \mathcal{R}^{n \times n}$ is a positive definite symmetric matrix, $p \in \mathcal{R}^n$, $\pi \in \mathcal{R}$, and $\tilde{P} = \begin{bmatrix} P & p \\ p^T & \pi \end{bmatrix}$. Then, the following corollary is easily shown.

Corollary 1: If \tilde{P} satisfies

- (i) $\tilde{A}^T \tilde{P} + \tilde{P} \tilde{A} = 0$,
- (ii) $\tilde{B}^T \tilde{P} + \tilde{P} \tilde{B} < 0$,

then any solution $x(t)$ of (3) except equilibrium solutions satisfies $\lim_{t \rightarrow \infty} V(x(t)) = 0$. \square

For a given \tilde{P} , we can construct a system with an asymptotically stable limit cycle satisfying $\tilde{x}^T \tilde{P} \tilde{x} = 0$ by choosing appropriate matrices \tilde{A} and \tilde{B} . It is noted that the matrix \tilde{A} is simply given by

$$\tilde{A} = \tilde{G}_A \tilde{P}, \quad \tilde{G}_A = \begin{bmatrix} G_A & 0 \\ 0 & 0 \end{bmatrix}, \quad (5)$$

where G_A is a nonzero skew-symmetric matrix. The matrix \tilde{B} can be also chosen by

$$\tilde{B} = \tilde{G}_B \tilde{P}, \quad \tilde{G}_B = \begin{bmatrix} G_B & 0 \\ 0 & 0 \end{bmatrix}, \quad (6)$$

where G_B is a matrix which satisfies $G_B^T + G_B < 0$.

Example 1 ($n = 2$): We consider the following symmetric matrix \tilde{P} :

$$\tilde{P} = \begin{bmatrix} 7.5 & 1 & -2.7 \\ 1 & 2.5 & 0 \\ -2.7 & 0 & -5 \end{bmatrix}. \quad (7)$$

Moreover, we set

$$\tilde{G}_A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (8)$$

$$\tilde{G}_B = \begin{bmatrix} -0.0462 & 0.002 & 0 \\ -0.0092 & -0.0314 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (9)$$

Then, by (5) and (6) we have

$$\tilde{A} = \begin{bmatrix} -1 & -2.5 & 0 \\ 7.5 & 1 & -2.7 \\ 0 & 0 & 0 \end{bmatrix}, \quad (10)$$

$$\tilde{B} = \begin{bmatrix} -0.3445 & -0.0412 & 0.1247 \\ -0.1004 & -0.0877 & 0.0248 \\ 0 & 0 & 0 \end{bmatrix}. \quad (11)$$

Example 2 ($n = 3$): Set \tilde{P} , \tilde{G}_A , and \tilde{G}_B as follows:

$$\tilde{P} = \begin{bmatrix} 2 & 0.8 & 1.4 & -1.7 \\ 0.8 & 1 & 0.3 & 2 \\ 1.4 & 0.3 & 3 & -0.5 \\ -1.7 & 2 & -0.5 & -3 \end{bmatrix}, \quad (12)$$

$$\tilde{G}_A = \begin{bmatrix} 0 & 1 & 0.5 & 0 \\ -1 & 0 & 1 & 0 \\ -0.5 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad (13)$$

$$\tilde{G}_B = \begin{bmatrix} -0.0185 & -0.0005 & 0.001 & 0 \\ 0.0015 & -0.0315 & -0.001 & 0 \\ -0.002 & -0.003 & -0.0005 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (14)$$

Then, we have

$$\tilde{A} = \begin{bmatrix} 1.5 & 1.15 & 1.8 & 1.75 \\ -0.6 & -0.5 & 1.6 & 1.2 \\ -1.8 & -1.4 & -1 & -1.15 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad (15)$$

$$\tilde{B} = \begin{bmatrix} -0.036 & -0.015 & -0.0231 & 0.03 \\ -0.0236 & -0.0306 & -0.0104 & -0.065 \\ -0.0071 & -0.0047 & -0.0052 & -0.0024 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (16)$$

Figures 1 and 2 show simulation results of Examples 1 and 2 from two initial states, respectively. Dashed lines in Figure 1 denote level curves of the Lyapunov function. Both solutions in this figure converge to the limit cycle which satisfies $V(x) = 0$. In contrast, in Figure 2, each solution converges to different limit cycles on an elliptic sphere which satisfy $V(x) = 0$.

It is clear from these results that this approach cannot synthesize a unique limit cycle when $n > 2$, because the constraint $V(x) = 0$ defines an $(n - 1)$ -dimensional manifold and solutions converge to different limit cycles on the manifold depending on the initial state. In order to synthesize a unique limit cycle in the case where $n > 2$, (3) is modified as the following system with a constraint function $V : \mathcal{R}^n \rightarrow \mathcal{R}^{n-1}$:

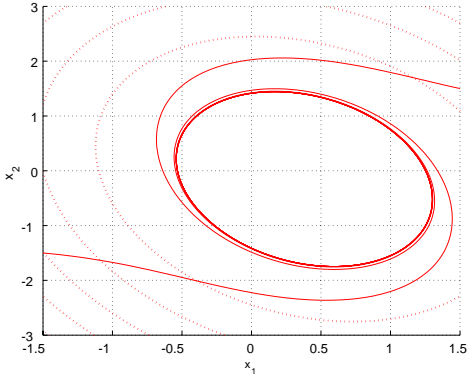


Fig. 1 Solutions of Example 1 from two initial conditions $x(0) = \pm[1.5 \ 1.5]^T$.

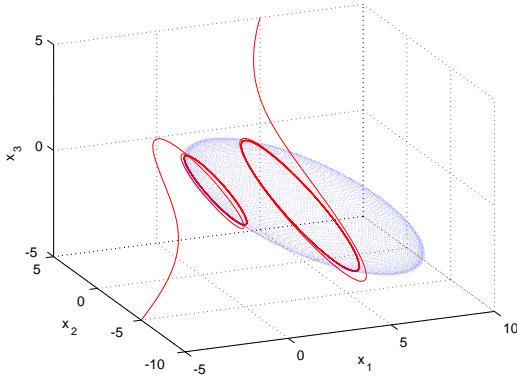


Fig. 2 Solutions of Example 2 from two initial conditions $x(0) = \pm[5 \ 5 \ 5]^T$.

$$\dot{\tilde{x}} = \begin{bmatrix} A & \mathbf{0} & a \\ \Xi_1 & \mathbf{0} & \xi_1 \\ \vdots & \vdots & \vdots \\ \Xi_{n-2} & \mathbf{0} & \xi_{n-2} \\ [0 \ 0] & \mathbf{0} & 0 \end{bmatrix} \tilde{x} + \begin{bmatrix} V_1 [B \ \mathbf{0} \ b] \tilde{x} \\ \alpha_1 V_2 \\ \vdots \\ \alpha_{n-2} V_{n-1} \\ 0 \end{bmatrix}, \quad (17)$$

where $\tilde{x} = [x_1 \ x_2 \ \dots \ x_n \ 1]^T$, $A, B \in \mathcal{R}^{2 \times 2}$, $a, b \in \mathcal{R}^{2 \times 1}$, $\Xi_i \in \mathcal{R}^{1 \times 2}$, $\xi_i, \alpha_i \in \mathcal{R}$, $i = 1, \dots, n-2$, and $V = [V_1 \ V_2 \ \dots \ V_{n-1}]^T : \mathcal{R}^n \rightarrow \mathcal{R}^{n-1}$. $V_i(x)$ is given by

$$V_i(x) = \begin{cases} [x_1 \ x_2 \ 1] \begin{bmatrix} P & p \\ p^T & \pi \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix}, & \text{if } i = 1, \\ \zeta_i x_1 + \eta_i x_2 + \lambda_i - x_{i+1}, & \text{otherwise.} \end{cases} \quad (18)$$

$V_1(x)$ defines an elliptic cylinder and the other functions $V_i(x)$, $i = 2, \dots, n-1$ define hyperplanes in the n -dimensional space. Figure 3 shows a configuration for $V_1(x)$ and $V_2(x)$. To construct a system (17) with an asymptotically stable limit cycle which satisfies $V(x) = 0$ (this defines a 1-dimensional manifold), we consider the condition $\frac{\partial V(x)}{\partial x} f(x) = 0$. The matrix A and the vector a can be determined by (5). For the other parameters Ξ_i, ξ_i , $i = 1, \dots, n-2$ must satisfy

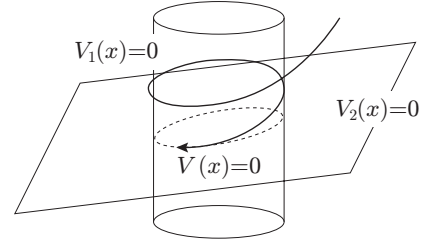


Fig. 3 The constraint $V(x) = 0$ in 3-dimensional space.

$$\frac{\partial V_i(x)}{\partial x} f(x) = [\zeta_i \ \eta_i \ -1] \begin{bmatrix} A & a \\ \Xi_{i-1} & \xi_{i-1} \end{bmatrix} = 0, \quad (19)$$

which implies

$$[\Xi_{i-1} \ \xi_{i-1}] = [\zeta_i \ \eta_i] [A \ a]. \quad (20)$$

Thus, using (5) and (20), we can construct the first term of (17) which has a desired limit cycle specified by the intersection of V_i . Next, we consider the second condition $\frac{\partial V_i(x)}{\partial x} g(x) V_i(x) < 0$. Here, we introduce the following proposition [4, Corollary 2.1].

Proposition 1: Assume that V_1 satisfies all conditions of Theorem 1. If all solutions of (1) are bounded, the second condition in Theorem 1 with respect to V_μ , $2 \leq \mu \leq n-1$, is equivalent to the following condition:

(ii)' For each μ th component of V , $2 \leq \mu \leq n-1$,

$$\frac{\partial V_\mu(x)}{\partial x} g(x) V_\mu(x) < 0,$$

$\forall x \in \Omega$ such that $V_1(x) = 0$. \square

By choosing the matrix B and the vector b from (6), it is guaranteed that all solutions converge to $V_1(x) = 0$ as $t \rightarrow \infty$. For any x satisfying $V_1(x) = 0$,

$$\begin{aligned} \frac{\partial V_i(x)}{\partial x} g(x) V_i(x) &= [\zeta_i \ \eta_i \ -1] \begin{bmatrix} V_1 [B \ \mathbf{0} \ b] \tilde{x} \\ \alpha_{i-1} V_i \end{bmatrix} V_i(x) \\ &= -\alpha_{i-1} V_i(x)^2 < 0. \end{aligned} \quad (21)$$

Hence, whenever B and b are determined, (17) satisfies the convergence condition $\frac{\partial V_i(x)}{\partial x} g(x) V_i(x) < 0$ from Proposition 1, and solutions converge to the limit cycle satisfying $V(x) = 0$. It is noted that $\alpha_{i-1} > 0$ represents a convergence rate of V_i .

Example 3: We consider the case when $V : \mathcal{R}^3 \rightarrow \mathcal{R}^2$. Let $V_1(x)$ be (7) and

$$V_2(x) = 1.5x_1 - 0.2x_2 - x_3. \quad (22)$$

We select matrices A, B and vectors a, b as (10) and (11). Using (20), we have

$$[\Xi_1 \ \xi_1] = [-3 \ -3.95 \ 0.54]. \quad (23)$$

Figure 4 shows a simulation result of this example with $\alpha_1 = 2$. Every solution converges to the same limit cycle.

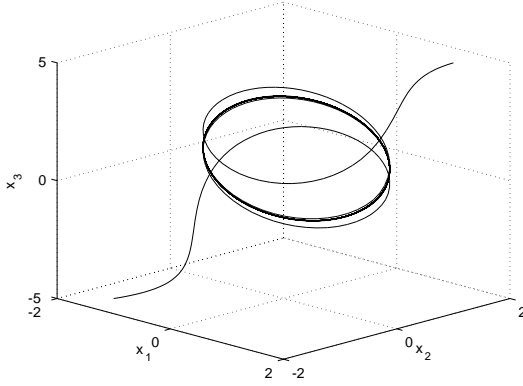


Fig. 4 Solutions of Example 3 from two initial conditions $x(0) = \pm[1.5 \ 1.5 \ 5]^T$.

3. Synthesis Method

We consider hybrid systems described by

$$\begin{cases} \dot{x}(t) = h(x(t), q(t)), \\ q^+(t) = \phi(x(t), q(t)), \end{cases} \quad (24)$$

where $x \in \mathcal{R}^n$ is the continuous state vector, $q \in Q = \{1, 2, \dots, M\}$ is the discrete state, $q^+(t)$ refers to the lefthand limit of $q(t)$ at time t , that is $q^+(t) = \lim_{\varepsilon \rightarrow +0} q(t + \varepsilon)$. The function $\phi : \mathcal{R}^n \times Q \rightarrow Q$ describes the change of discrete states, and a switching of discrete states from q to r is defined by a switch set

$$S_{q,r} = \{x \in \mathcal{R}^n \mid \phi(x, q) = r\}, \quad q, r \in Q. \quad (25)$$

The function $h : \mathcal{R}^n \times Q \rightarrow \mathcal{R}^n$ shows a vector field, and the continuous state $x(\cdot)$ evolves according to $h(\cdot, q)$ for each state $q \in Q$. In this paper, each $h(\cdot, q)$ is called a subsystem q .

Definition 1: A solution $(x(t), q(t))$ of (24) is said to be well-defined if the following conditions hold:

- (i) The solution is defined for $t \in [0, \infty)$.
- (ii) There exists a finite or infinite sequence $\{t_n\}_{n=0}^N$ such that $t_0 = 0$, $t_{n+1} > t_n$, $n = 0, 1, 2, \dots$, $\lim_{n \rightarrow N} t_n = \infty$. $q(t)$ is discontinuous only at t_n and constant for $t_n \leq t < t_{n+1}$.

$\{t_n\}_{n=0}^N$ is called a switching sequence of the solution $(x(t), q(t))$.

The hybrid space of (24) is given by $\mathcal{H} := \mathcal{R}^n \times Q$. Consider an initial state which lies in a set of possible initial conditions $(x_0, q_0) \in \mathcal{H}_0 \subset \mathcal{H}$, and assume that a solution $(x(t), q(t))$ starting from (x_0, q_0) is well-defined. The solution of (24) evolves according to $\dot{x} = h(x, q_0)$, and if a state $x(t)$ hits a switch set $S_{q,r}$ at time t_n , the corresponding discrete transition from the discrete state q to r occurs. The evolution of the discrete state can be described by a sequence

$$\xi(x_0, q_0) = (q_0, t_0), (q_1, t_1), \dots, \quad (26)$$

where (q_k, t_k) means that $\dot{x} = h(x(t), q_k)$ for $t_k \leq t < t_{k+1}$ and $q^+(t_{k+1}) = \phi(x(t_{k+1}), q_k) = q_{k+1}$. For (26), we define a projection to a time sequence:

$$\xi_t(x_0, q_0) = t_0, t_1, t_2, \dots \quad (27)$$

To express a sequence of the time interval where discrete state equals q , we define the following projection:

$$\xi_t(x_0, q_0) | q = t_0^q, t_1^q, \dots, t_{2k}^q, t_{2k+1}^q, \dots \quad (28)$$

where t_{2k}^q and t_{2k+1}^q are time instances where the subsystem q is switched on and off, respectively. Furthermore, to obtain the duration which the system is driven by the subsystem q , we define the interval completion $I(\xi_t(x_0, q_0) | q)$ as a set obtained by taking the union of all intervals

$$I(\xi_t(x_0, q_0) | q) = \bigcup_{k \in \mathbb{N}} [t_{2k}^q, t_{2k+1}^q]. \quad (29)$$

Let $E(\xi_t(x_0, q_0) | q)$ be the even sequence of $\xi_t(x_0, q_0) | q$:

$$E(\xi_t(x_0, q_0) | q) = t_0^q, t_2^q, \dots, t_{2k}^q, \dots \quad (30)$$

The conditions of Theorem 1 are based on a smooth constraint equation. Using a piecewise smooth constraint equation, we will extend Theorem 1 to hybrid systems given by

$$\begin{cases} \dot{x}(t) = f(x(t), q(t)) + g(x(t), q(t)), \\ q^+(t) = \phi(x(t), q(t)). \end{cases} \quad (31)$$

Theorem 2: If there exists a continuously differentiable function $\mathcal{V}_q : \Omega_q \rightarrow \mathcal{R}^m$, for all $q \in Q$ where Ω_q is a subset of \mathcal{R}^n , ($n > m$) such that

- (i) $\frac{\partial \mathcal{V}_q(x)}{\partial x} f(x, q) = 0, \quad \forall x \in \Omega_q, \forall t \in I(\xi_t(x_0, q_0) | q)$,
- (ii) For each μ th component of $\mathcal{V}_q, 1 \leq \mu \leq m$,

$$\frac{\partial \mathcal{V}_{q_\mu}(x)}{\partial x} g(x, q) \mathcal{V}_{q_\mu}(x) < 0,$$

- (iii) $\mathcal{V}_q(x) = \mathcal{V}_r(x), \quad x \in S_{q,r}, \forall t \in E(\xi_t(x_0, q_0) | r)$,

then any solution $(x(t), q(t)) \in \Omega_{q(t)}$ of (31) except equilibrium solutions of each subsystem satisfies $\lim_{t \rightarrow \infty} \mathcal{V}_{q(t)}(x(t)) = 0$. \square

Proof: Due to the first and second condition, it guarantees that $\|\mathcal{V}_q(x(t))\|$ decreases for all $t \in I(\xi_t(x_0, q_0) | q)$ from Theorem 1. From the third condition, for any discrete transition from q to q' at $\bar{t} \in E(\xi_t(x_0, q_0) | q')$, we have $\mathcal{V}_q(x(\bar{t})) = \mathcal{V}_{q'}(x(\bar{t}))$. Consequently, we conclude $\|\mathcal{V}_{q(t)}(x(t))\|$ decreases for all $t \in \cup_{q \in Q} I(\xi_t(x_0, q_0) | q)$ and all solutions $(x(t), q(t))$ of (31) satisfy $\mathcal{V}_{q(t)}(x(t)) = 0$ as $t \rightarrow \infty$. \square

It is noted that the convergence of $\|\mathcal{V}_{q(t)}(x(t))\|$ is assured even if there exist \hat{q} whose interval completion $I(\xi_t(x_0, q_0) | \hat{q})$ ends up with finite time, because

the third condition of Theorem 2 requires the continuity of $\mathcal{V}_{q(t)}(x(t))$ with respect to all possible discrete transitions, that is, all possible switching sequences.

We consider the following hybrid system with \mathcal{V}_{q_i} given by (18).

$$\left\{ \begin{array}{l} \dot{\tilde{x}} = \begin{bmatrix} A_q & \mathbf{0} & a_q \\ \Xi_{q_1} & \mathbf{0} & \xi_{q_1} \\ \vdots & \vdots & \vdots \\ \Xi_{q_{n-2}} & \mathbf{0} & \xi_{q_{n-2}} \\ [0 \ 0] & \mathbf{0} & 0 \end{bmatrix} \tilde{x} + \begin{bmatrix} \mathcal{V}_{q_1} [B_q \ \mathbf{0} \ b_q] \tilde{x} \\ \alpha_{q_1} \mathcal{V}_{q_2} \\ \vdots \\ \alpha_{q_{n-2}} \mathcal{V}_{q_{n-1}} \\ 0 \end{bmatrix}, \\ q^+(t) = r, \quad \text{if } q(t) = q \text{ and } x(t) \in S_{q,r}, \end{array} \right. \quad (32)$$

where the transition of discrete state from q to r occurs whenever solutions hit a hyperplane

$$S_{q,r} = \{x \in \mathcal{R}^n \mid \tilde{c}_{q,r}^T \tilde{x} = 0\}, \quad q, r \in Q, \quad (33)$$

where $\tilde{c}_{q,r} = [c_{q,r}^T \ d_{q,r}]^T$, $c_{q,r} \in \mathcal{R}^n$, and $d_{q,r} \in \mathcal{R}$. For given \mathcal{V}_q , this hybrid system satisfies both the first and second condition in Theorem 2 by determining each parameter $A_q, a_q, B_q, b_q, \Xi_q$ and ξ_q from (5), (6), (20). But \mathcal{V}_q is not allowed to choice freely because of the third condition. This condition requires the continuity of constraint functions on all switching surfaces, and it is hard to find such constraint functions in general. From Proposition 1, however, if \mathcal{V}_{q_1} satisfies the continuity on all switching surfaces, the second and third condition of Theorem 2 is relaxed.

Since \mathcal{V}_{q_1} is quadratic, this type of quadratic functions can be formulated by piecewise quadratic Lyapunov functions using the conditions for discrete transitions [7]. Let the quadratic function be $\mathcal{V}_{q_1}(x) = \tilde{x}^T \tilde{P}_q \tilde{x}$ and consider the case that the discrete state switches from q to r . The condition of switching is given by $\tilde{c}_{q,r}^T \tilde{x} = 0$. Since \mathcal{V}_{q_1} should be continuous on switching surfaces, the third condition in Theorem 2 can be formulated by

$$\tilde{P}_r = \tilde{P}_q + \tilde{t}_{q,r}^T \tilde{c}_{q,r} + \tilde{c}_{q,r}^T \tilde{t}_{q,r}, \quad (34)$$

where $\tilde{t}_{q,r}$ is an $(n+1)$ -dimensional vector.

In a piecewise affine system which is a special class of hybrid systems, the state space is partitioned into several regions $X_q \subseteq \mathcal{R}^n$, $q \in Q$, and each region corresponds to a discrete state. If each region forms a polyhedron with pairwise disjoint interior, we can obtain matrices $\tilde{E}_q = [E_q \ e_q]$, $\tilde{F}_q = [F_q \ f_q]$ such that

$$\tilde{E}_q \tilde{x} \geq 0, \quad x \in X_q, \quad (35)$$

$$\tilde{F}_q \tilde{x} = \tilde{F}_r \tilde{x}, \quad x \in X_q \cap X_r. \quad (36)$$

In this paper, these polyhedrons are given by

$$\tilde{E}_q = [E_{q_1} \ E_{q_2} \ 0 \ \cdots \ 0 \ e_q], \quad (37)$$

where E_{q_i} is the i th column vector of E_q , since we consider a two-dimensional quadratic function as \mathcal{V}_{q_1} independently of the dimension of the system. By using this representation, the requirement that \mathcal{V}_{q_1} is continuous at every point on the switching surface can be written as

$$\tilde{P}_q = \tilde{F}_q T \tilde{F}_q, \quad (38)$$

where T is a symmetric matrix. Then the continuity of \mathcal{V}_{q_1} is assured by using (34) or (38), from Proposition 1 and we have the following proposition.

Proposition 2: Assume that \mathcal{V}_{q_1} satisfies all conditions of Theorem 2 for each $q \in Q$. If all solutions of (32) are well-defined, the second and third conditions of Theorem 2 with respect to \mathcal{V}_{q_μ} , $2 \leq \mu \leq n-1$, are equivalent to the following conditions:

(ii)' For each μ th component of \mathcal{V}_q , $2 \leq \mu \leq n-1$,

$$\frac{\partial \mathcal{V}_{q_\mu}(x)}{\partial x} g(x, q) \mathcal{V}_{q_\mu}(x) < 0,$$

$$\forall x \in \Omega_q \text{ s.t. } \mathcal{V}_{q_1}(x) = 0, \forall t \in I(\xi_t(x_0, q_0) | q).$$

(iii)' For each μ th component of \mathcal{V}_q , $2 \leq \mu \leq n-1$,

$$\mathcal{V}_{q_\mu}(x) = \mathcal{V}_{r_\mu}(x), \quad x \in S_{q,r} \text{ s.t. } \mathcal{V}_{q_1}(x) = \mathcal{V}_{r_1}(x), \\ \forall t \in E(\xi_t(x_0, q_0) | r). \quad \square$$

Example 4: We consider the following piecewise smooth constraint equation $\mathcal{V}_q(x) = 0$, $q \in \{1, 2, 3, 4\}$ as follows:

$$\mathcal{V}_1(x) = \begin{bmatrix} 6x_1^2 + 2x_2^2 - 8 \\ x_2 - x_3 \end{bmatrix}, \quad \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \end{bmatrix} x \geq 0,$$

$$\mathcal{V}_2(x) = \begin{bmatrix} 2x_1^2 + 6x_2^2 - 8 \\ x_1 - x_3 \end{bmatrix}, \quad \begin{bmatrix} 1 & -1 & 0 \\ -1 & -1 & 0 \end{bmatrix} x \geq 0,$$

$$\mathcal{V}_3(x) = \begin{bmatrix} 6x_1^2 + 2x_2^2 - 8 \\ -x_2 - x_3 \end{bmatrix}, \quad \begin{bmatrix} -1 & 1 & 0 \\ -1 & -1 & 0 \end{bmatrix} x \geq 0,$$

$$\mathcal{V}_4(x) = \begin{bmatrix} 2x_1^2 + 6x_2^2 - 8 \\ -x_1 - x_3 \end{bmatrix}, \quad \begin{bmatrix} -1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} x \geq 0,$$

Then, \mathcal{V}_{q_1} , $q \in \{1, 2, 3, 4\}$, are continuous on all switching surfaces, and \mathcal{V}_{q_2} are also continuous under the condition $\mathcal{V}_{q_1} = \mathcal{V}_{r_1}$. Thus we can construct the following hybrid system from (32):

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 2x_2 - (6x_1^2 + 2x_2^2 - 8)x_1 \\ -6x_1 - (6x_1^2 + 2x_2^2 - 8)x_2 \\ -6x_1 + (x_2 - x_3) \end{bmatrix}, \quad \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \end{bmatrix} x \geq 0,$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 6x_2 - (2x_1^2 + 6x_2^2 - 8)x_1 \\ -2x_1 - (2x_1^2 + 6x_2^2 - 8)x_2 \\ 6x_2 + (x_1 - x_3) \end{bmatrix}, \quad \begin{bmatrix} 1 & -1 & 0 \\ -1 & -1 & 0 \end{bmatrix} x \geq 0,$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 2x_2 - (6x_1^2 + 2x_2^2 - 8)x_1 \\ -6x_1 - (6x_1^2 + 2x_2^2 - 8)x_2 \\ 6x_1 + (-x_2 - x_3) \end{bmatrix}, \quad \begin{bmatrix} -1 & 1 & 0 \\ -1 & -1 & 0 \end{bmatrix} x \geq 0,$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 6x_2 - (2x_1^2 + 6x_2^2 - 8)x_1 \\ -2x_1 - (2x_1^2 + 6x_2^2 - 8)x_2 \\ -6x_2 + (-x_1 - x_3) \end{bmatrix}, \quad \begin{bmatrix} -1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} x \geq 0,$$

Every solution except equilibrium solutions converges to the desired limit cycle as shown in Figure 5.

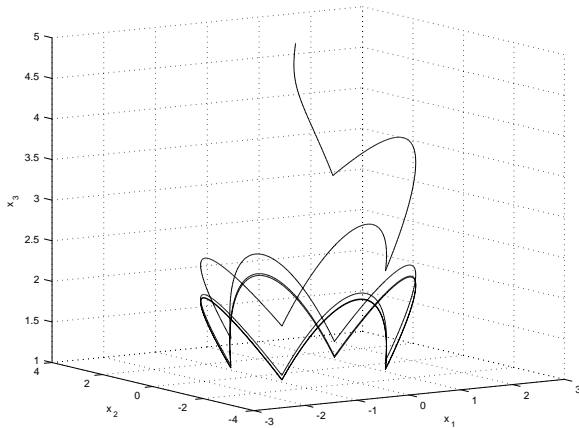


Fig. 5 Solution of Example 4 from initial condition $x(0) = [0 \ 0.5 \ 5]^T$.

4. Conclusions

In this paper, we propose a synthesis method of hybrid systems with limit cycles, which are given by a piecewise smooth constraint equation $\mathcal{V}_q(x) = 0$. Limit cycles of designed hybrid systems are composed by elliptic cylinders and hyperplanes. Synthesis of hybrid systems with more general form of limit cycles is a future study.

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Masakazu Adachi received B.E. and M.E. degrees in 2002, 2003, respectively, from Osaka University, where he is currently studying for the Ph.D. degree. His research interest includes hybrid systems.

Toshimitsu Ushio received B.S., M.S. and Ph.D. degrees in 1980, 1982, 1985, respectively, from Kobe University. He joined Osaka University as an Associate Professor in 1994, and is currently a professor. His research interests include nonlinear oscillation and control of discrete event systems. He is a member of SICE, ISCIE, and IEEE.

Shigeru Yamamoto received B.E., M.E. and Ph.D. degrees in 1987, 1989, 1996, respectively, from Osaka University. He has been with the Department of Systems and Human Science, Osaka University, since 1998, where he is currently an Associate Professor. His research interests include hybrid systems, and controlling chaos. He is a member of SICE, ISCIE, IEEE, and JSME.