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Application of Lyapunov Function Based Synthesis of Nonsmooth Limit Cycles to Motion Generation for Humanoid Robots

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Abstract

Periodic motion patterns of a humanoid robot can be characterized by several closed curves in a high dimensional space corresponding to the degree-of-freedom of the robot. Whole body motions are expressed by combining and interacting these patterns. However, it is difficult to select suitable motion dynamically. Both synthesis of nonlinear systems with a limit cycle corresponding to each periodic motion and suitable transitions between periodic motions are very useful to realize adaptive behaviors according to environments. In this paper, we propose a Lyapunov function based method of periodic motion generation and a method of dynamic transition between motions of humanoid robots. The transient behavior between motions is achieved by destabilizing the current motion and stabilizing the next motion.

Keywords: Motion generation, Whole body motion, Lyapunov function, Nonlinear dynamics

1. Introduction

Flexible motion generation of humanoid robots is one of the most challenging problems, and many approaches have been proposed [1, 2]. Among them, application of nonlinear dynamics is useful for representing periodic motions such as gaits. For example, a central pattern generator is a functional neural network which produces a coordinated activities in legged animals, and its mathematical model is applied to humanoid robots [3]. In order to apply nonlinear dynamics to motion generator of humanoid robots, we have two problems: one is representation (symbolization) of typical periodic motions in a reduced state space since humanoid robots have large number of degree-of-freedom. The other is a mechanism of dynamic change of periodic motions in order to adapt their behaviors to environments.

For the representation problem of periodic motions, many methods have been proposed. Tatani and Nakamura showed a possibility of representation of motion patterns [4]. They obtained common space of a motion pattern by hierarchically arranged neural networks which can reduce the dimension of motion effectively. Okada *et al.* also showed a reduction method of the whole body motion based on the principal component analysis using singular value decomposition [5].

For the second problem, Sekiguchi and Nakamura developed the behavior control of robots by using nonlinear phenomena, such as entrainment and synchronization [6]. Dynamic changes of periodic motions are also achieved by switching of vector fields [5]. Obviously, transient behaviors from one periodic motion to another are closely related to switching procedure of vector fields so that the current periodic motion are destabilized and a desired periodic motion is stabilized. But there are few studies for taking into consideration how to change the vector fields.

This paper propose a Lyapunov function based synthesis of motion generation of humanoid robots. In the proposed method, we calculate a Lyapunov function which keeps constant on a periodic motion and obtain a desired system. The designed system has a stable limit cycle corresponding to a periodic motion, and it is easy to change the stability of the limit cycle. Our proposed method is based on [7, 8, 9, 10].

This paper is organized as follows. In Section 2, we revisit some useful results reported in [10], and show a designing method of the system which generates a smooth periodic motion. In Section 3, we introduce piecewise quadratic Lyapunov functions, and generation of nonsmooth periodic motions is discussed. In Section 4, we design the humanoid whole body motion by connecting the obtained systems.

2. Generation of Smooth Motions

In this section, we first review a design method of nonlinear system with a smooth motion pattern proposed by Green [10].

2.1. Preliminary

We consider the following continuous differential equations:

$$\dot{x} = f(x) + g(x), \tag{1}$$

where $f : \mathbf{R}^n \to \mathbf{R}^n, g : \mathbf{R}^n \to \mathbf{R}^n$.

Green shows sufficient conditions for the trajectories of (1) converge to a limit cycle satisfying a given constraint $V(x) = 0$, where $V : \mathbf{R}^n \to \mathbf{R}^m$ is continuously differentiable.

Theorem 1 (Green [10]) *If there exists a continuously differentiable function $V : \Omega \to \mathbf{R}^m$ where Ω is a subset of $\mathbf{R}^n, n > m$, such that*

- $\dfrac{\partial V(x)}{\partial x} f(x) = 0, \quad \forall x \in \Omega$
- *For each μth component of $V, 1 \le \mu \le m$,*
  $$\frac{\partial V_\mu(x)}{\partial x} g(x) V_\mu(x) < 0,$$
  $$\forall x \in \Omega \text{ such that } V_\mu(x(t)) \neq 0.$$

Then, all trajectories of (1) converge to $V(x) = 0$ as $t \to \infty$.

Corollary 1 (Green [10]) *Let $V_a : \mathbf{R}^n \to \mathbf{R}^{m_a}, V_b : \mathbf{R}^n \to \mathbf{R}^{m_b}, m_a + m_b = m$. Assume that V_a satisfies the conditions of Theorem 1. If all solutions of (1) with initial conditions in Ω are bounded, then the second condition of Theorem 1 is rewritten as follows:*

- *For each μth component of $V_b, 1 \le \mu \le m_b$,*
  $$\frac{\partial V_\mu(x)}{\partial x} g(x) V_\mu(x) < 0,$$
  $$\forall x \in \Omega \text{ such that } V_a(x) = 0.$$

If there exists a Lyapunov function $V(x)$ which satisfies Theorem 1 for (1), a trajectory starting from any initial point converges to the hypersurface $V(x) = 0$, and the steady state forms a closed curve on this hypersurface.

Here, it is noted that an arbitrary periodic solution in n-dimensional space can be expressed by using Fourier series and Chebyshev polynomials as follows :

$$\begin{aligned} y(t) &= \sum_{k=0}^{\infty} \Big\{ \alpha_k \cos(k\omega t) + \beta_k \sin(k\omega t) \Big\} \\ &= \sum_{k=0}^{\infty} \Big\{ a_k T_k(\cos(\omega t)) + b_k \sin(\omega t) U_k(\cos(\omega t)) \Big\} \\ &= F_1(\cos(\omega t)) + \sin(\omega t) F_2(\cos(\omega t)), \end{aligned} \tag{2}$$

where $y(t)$, α_k and $\beta_k \in \mathbf{R}^n$, a_k and b_k depend on α_k and β_k, T_k and U_k are the kth Chebyshev polynomials of the first and second kind respectively, $F_1, F_2 : \mathbf{R} \to \mathbf{R}^n$.

From Theorem 1 and Corollary 1, if (1) is restricted to the following form:

$$\begin{aligned} \dot{x} &= f(x) + g(x) \\ &= \omega \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ \dfrac{-\partial F_1(x_2)}{\partial x_2} - x_1 \dfrac{F_2(x_2)}{\partial x_2} & F_2(x_2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &\quad + \alpha \begin{bmatrix} x_1(1 - x_1^2 - x_2^2) \\ x_2(1 - x_1^2 - x_2^2) \\ F_1(x_2) + x_1 F_2(x_2) - y \end{bmatrix}, \end{aligned} \tag{3}$$

where $\alpha > 0$ represents the convergence rate. Then there exists the following Lyapunov function:

$$V(x) = \begin{bmatrix} x_1^2 + x_2^2 - 1 \\ F_1(x_2) + x_1 F_2(x_2) - y \end{bmatrix}. \tag{4}$$

All trajectories of this differential equation converge to the hypersurface $V(x) = 0$, i.e., $x_1(t) = \sin(\omega t)$, $x_2(t) = \cos(\omega t)$ and $y(t) = F_1(x_2(t)) + x_1(t) F_2(x_2(t))$ respectively as $t \to \infty$. This result shows that this system can generate an arbitrary asymptotically stable limit cycle.

2.2. Design of the system

Consider the humanoid robots with n degree-of-freedom. For a periodic motion like gait, for example, the motion data M is given as follows:

$$M = \begin{bmatrix} \theta_1[t_1] & \theta_1[t_2] & \cdots & \theta_1[t_T] \\ \theta_2[t_1] & \theta_2[t_2] & \cdots & \theta_2[t_T] \\ \vdots & \vdots & & \vdots \\ \theta_n[t_1] & \theta_n[t_2] & \cdots & \theta_n[t_T] \end{bmatrix}, \tag{5}$$

where $\theta_i[t_j]$ is the ith joint angle at the time t_j. Generally, it is difficult to treat the motion data M without any reduction because the humanoid robot has high degree-of-freedom. So first, we reduce the motion data M to a lower-dimensional data. By applying a reduction method using singular value decomposition [5], the reduced data is obtained as follows:

$$\begin{aligned} M &= USV^T, \\ U &= \left[\, U_1 \mid U_2 \,\right], \\ S &= \mathrm{blockdiag}\{S_1\ S_2\}, \\ S_1 &= \mathrm{diag}\{s_1\ s_2\ \ldots\ s_m\}, \\ S_2 &= \mathrm{diag}\{s_{m+1}\ s_{m+2}\ \ldots\ s_n\}, \\ V &= \left[\, V_1 \mid V_2 \,\right], \end{aligned}$$

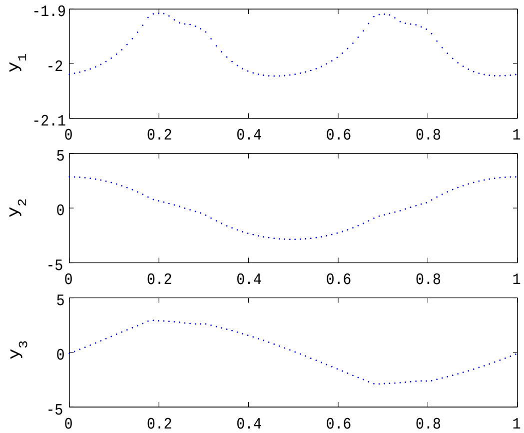

Figure 1: Time sequence of reduced “walk” motion.

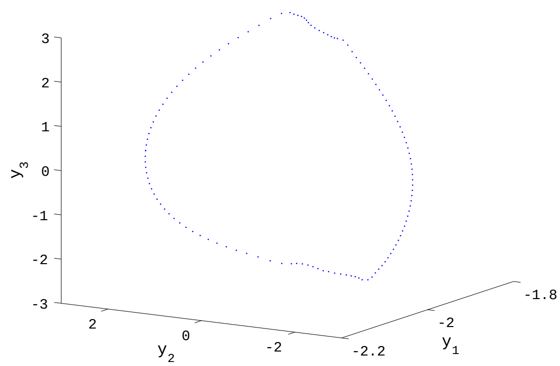

Figure 2: Closed curve of the reduced “walk” motion.

if $s_m \gg s_{m+1}$ is satisfied, the motion data M is reduced to m-dimensional data V_1^T as follows:

$$M = U_1 S_1 V_1^T. \tag{6}$$

In this paper, V_1^T is called a reduced motion data Y.

Figures 1 and 2 show a time sequence of reduced “walk” motion data $Y_{\mathrm{w}} \in \mathbf{R}^3$ and its closed curve. From (2), we approximate the reduced data Y as follows:

$$y(t) = \sum_{k=0}^{l} \left\{ \alpha_k \cos(k\omega t) + \beta_k \sin(k\omega t) \right\} \tag{7}$$

where l is an appropriate constant. Then, we have

$$\Lambda = (\Phi^T \Phi)^{-1} \Phi^T Y^T \tag{8}$$

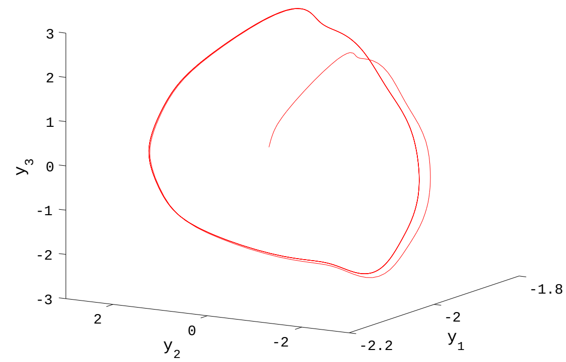

Figure 3: Trajectory of the designed system with reduced “walk” motion.

where

$$\Lambda = \begin{bmatrix} \alpha_0 & \alpha_1 & \cdots & \alpha_l & \beta_1 & \cdots & \beta_l \end{bmatrix}^T,$$

$$\Phi = \begin{bmatrix} 1 & \cos(t_1) & \cdots & \cos(lt_1) & \sin(t_1) & \cdots & \sin(lt_1) \\ 1 & \cos(t_2) & \cdots & \cos(lt_2) & \sin(t_2) & \cdots & \sin(lt_2) \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 1 & \cos(t_T) & \cdots & \cos(lt_T) & \sin(t_T) & \cdots & \sin(lt_T) \end{bmatrix}.$$

Thus, we get the function $y(t)$ which is an approximation of Y

$$y(t) = F_1(\cos(\omega t)) + \sin(\omega t) F_2(\cos(\omega t)). \tag{9}$$

By substituting (9) into (3), we obtain the desired system. Figure 3 shows a trajectory of the designed system which is approximated by $l = 9$. When l is sufficiently large, this system generates a asymptotically stable limit cycle corresponding to the closed curve of the reduced motion data Y.

3. Generation of Nonsmooth Motions

The motion of humanoid robots is subject to geometric constraints. These constraints cause the nonsmoothness of motion. However, the method proposed in the previous section does not generate a nonsmooth limit cycles. In this section, we propose a synthesis of system which generates a nonsmooth periodic motion. For simplicity, to generate a nonsmooth motion we use a piecewise quadratic Lyapunov function [11].

We split the reduced motion data Y into several

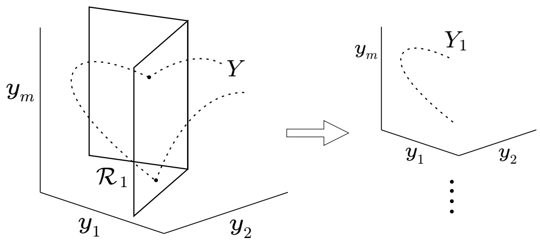

Figure 4: Splitting a nonsmooth closed curve into some ellipsoidal curves.

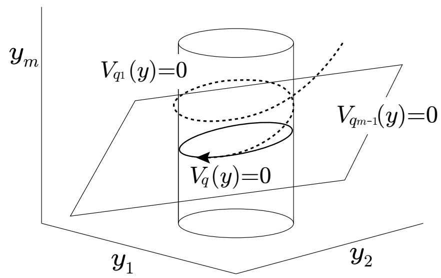

Figure 5: An ellipsoidal curve defined by an elliptic cylinder and a hyperplane.

components as follows:

$$Y = \left[\ Y_1 \,\middle|\, Y_2 \,\middle|\, \cdots \,\middle|\, Y_Q\ \right] = \left[\begin{array}{cc|c|cc} y_1[t_1] & \cdots & \cdots & \cdots & y_1[t_T] \\ y_2[t_1] & \cdots & \cdots & \cdots & y_2[t_T] \\ \vdots & & & & \vdots \\ y_m[t_1] & \cdots & \cdots & \cdots & y_m[t_T] \end{array}\right]. \quad (10)$$

The reduced space is split into several polyhedral regions whose boundaries are given by hyperplanes, and the periodic motions are represented by an ellipsoidal curve in each region $\mathcal{R}_q$ as shown in Figure 4.

By constructing subsystems corresponding to each ellipsoidal curve and switching those subsystems, nonsmooth motion pattern can be realized. Assume that each component $Y_q, q = 1, \ldots, Q$ can be approximated by an elliptic cylinder and a hyperplane (Figure 5). Hence, the approximated ellipsoidal curve is given by $V_q(y) = 0, V_q : \mathbf{R}^m \to \mathbf{R}^{m-1}$

$$V_q = \begin{bmatrix} V_{q_1} \\ V_{q_2} \\ \vdots \\ V_{q_{m-1}} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} y_1 & y_2 & 1 \end{bmatrix} \begin{bmatrix} P_q & p_q \\ p_q & \pi \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ 1 \end{bmatrix} \\ \zeta_{q_2} y_1 + \eta_{q_2} y_2 + \lambda_{q_2} - y_3 \\ \vdots \\ \zeta_{q_{m-1}} y_1 + \eta_{q_{m-1}} y_2 + \lambda_{q_{m-1}} - y_m \end{bmatrix} \quad (11)$$

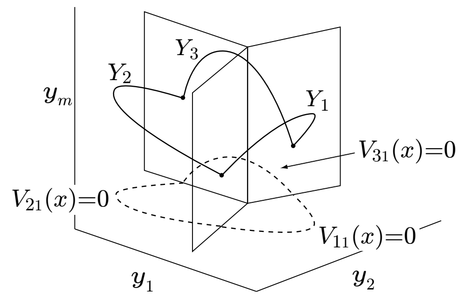

Figure 6: Piecewise quadratic Lyapunov function V_{q_1}.

where P_q is a positive definite matrix.

Here, in order to guarantee the stability of limit cycle in this switched system, as reported in [12], $V_{q_1}(y)$ is not allowed to choose freely when subsystems are combined. The requirement is the continuity of Lyapunov functions $V_{q_1}(y)$ on all region boundaries (Figure 6). The boundary between $\mathcal{R}_q$ and $\mathcal{R}_r$ is given by

$$\tilde{c}_{q,r}^T \begin{bmatrix} y_1 \\ y_2 \\ 1 \end{bmatrix} = 0, \quad (12)$$

where $\tilde{c}_{q,r} = \begin{bmatrix} c_{q,r}^T & d_{q,r} \end{bmatrix}^T$, and $c_{q,r} \in \mathbf{R}^2$, $d_{q,r} \in \mathbf{R}$. The Lyapunov function $V_{q_1}(y)$ is piecewise quadratic if and only if, for some $\tilde{t}_{q,r} \in \mathbf{R}^3$,

$$\tilde{P}_r = \tilde{P}_q + \tilde{t}_{q,r}^T \tilde{c}_{q,r} + \tilde{c}_{q,r}^T \tilde{t}_{q,r}, \quad (13)$$

where $\tilde{P}_q = \begin{bmatrix} P_q & p_q \\ p_q^T & \pi \end{bmatrix}$. If each region forms a polyhedron with pairwise disjoint interior, we can obtain the following matrices for each region:

$$\tilde{E}_q = \begin{bmatrix} E_q & e_q \end{bmatrix}, \ \ \tilde{F}_q = \begin{bmatrix} F_q & f_q \end{bmatrix},$$

where

$$\tilde{E}_q \begin{bmatrix} y_1 \\ y_2 \\ 1 \end{bmatrix} \geq 0, \quad y \in \mathcal{R}_q, \quad (14)$$

$$\tilde{F}_q \begin{bmatrix} y_1 \\ y_2 \\ 1 \end{bmatrix} = \tilde{F}_r \begin{bmatrix} y_1 \\ y_2 \\ 1 \end{bmatrix}, \quad y \in \mathcal{R}_q \cap \mathcal{R}_r. \quad (15)$$

By using this representation, the requirement that a Lyapunov function is continuous at every point on the region boundary can be written as

$$\tilde{P}_q = \tilde{F}_q T \tilde{F}_q, \quad (16)$$

where T is a symmetric matrix.

For ellipsoidal curves $V_q(y) = 0$, $q = 1, \dots, Q$ approximated under the condition (13) or (16), we can synthesize the following switched system whose trajectories converge to a nonsmooth periodic motion:

$$\begin{aligned}\dot{y} &= f(y) + g(y) \\ &= \begin{bmatrix} A_q & a_q \\ \Xi_{q_1} & \xi_{q_1} \\ \vdots & \vdots \\ \Xi_{q_{m-2}} & \xi_{q_{m-2}} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ 1 \end{bmatrix} + \alpha \begin{bmatrix} V_{q_1} \begin{bmatrix} B_q b_q \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ 1 \end{bmatrix} \\ V_{q_2} \\ \vdots \\ V_{q_{m-1}} \end{bmatrix} \\ & \qquad \text{if} \quad y \in \mathcal{R}_q. \end{aligned} \tag{17}$$

where $A_q, B_q \in \mathbf{R}^{2\times 2}$, $a_q, b_q \in \mathbf{R}^{2\times 1}$, $\Xi_{q_i} \in \mathbf{R}^{1\times 2}$, $\xi_{q_i} \in \mathbf{R}$, $i = 1, \dots, m-2$. Here A_q, a_q, B_q and b_q are given by

$$\tilde{A}_q \tilde{P}_q + \tilde{P}_q \tilde{A}_q = 0, \tag{18}$$
$$\tilde{B}_q \tilde{P}_q + \tilde{P}_q \tilde{B}_q < 0, \tag{19}$$

where $\tilde{A}_q = \begin{bmatrix} A_q & a_q \\ 0 & 0 \end{bmatrix}$, $\tilde{B}_q = \begin{bmatrix} B_q & b_q \\ 0 & 0 \end{bmatrix}$. We can simply choose these matrices as follows:

$$\tilde{A}_q = \tilde{G}_{A_q} \tilde{P}_q, \quad \tilde{G}_{A_q} = \begin{bmatrix} G_{A_q} & 0 \\ 0 & 0 \end{bmatrix}, \tag{20}$$

where G_{A_q} is an arbitrary skew-symmetric matrix. The matrix $\tilde{B}_q$ can be chosen as follows:

$$\tilde{B}_q = \tilde{G}_{B_q} \tilde{P}_q, \quad \tilde{G}_{B_q} = \begin{bmatrix} G_{B_q} & 0 \\ 0 & 0 \end{bmatrix}, \tag{21}$$

where G_{B_q} is a matrix which satisfies $G_{B_q}^T + G_{B_q} < 0$. Furthermore, Ξ_{q_i} and ξ_{q_i} are determined by

$$\begin{bmatrix} \Xi_{q_i} & \xi_{q_i} \end{bmatrix} = \begin{bmatrix} \zeta_{q_i} & \eta_{q_i} \end{bmatrix} \begin{bmatrix} A_q & a_q \end{bmatrix}. \tag{22}$$

Figure 7 shows a trajectory of designed system corresponding to reduced "squat" motion $Y_s \in \mathbf{R}^3$. In the proposed method, how to divide the motion data Y depend on the designer. In Figure 7, we simply divide motion data "squat" into two parts. It is clear that the approximation becomes accurate when we split the motion data into a large number of components. The restriction (13) or (16), however, becomes tight.

4. Design of Whole Body Motions

In this section, we design the humanoid whole body motion. It is clear that designed system $\dot{y} = f(y) + g(y)$ can be easily destabilized by changing $g(x)$ to $-g(x)$. When we consider whole body motion in which there exists multiple motion patterns this property is useful for trajectories to diverge from the current motion and converge to the next motion.

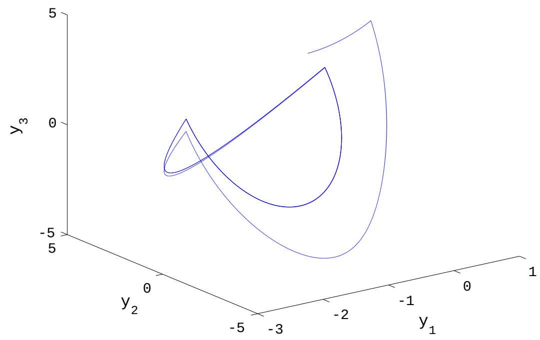

Figure 7: Trajectory of the designed system with reduced "squat" motion.

We consider the systems Σ and $\hat{\Sigma}$ with stable and unstable limit cycle (periodic motion), respectively.

$$\Sigma \;:\; \dot{y} = f(y) + g(y), \tag{23}$$
$$\hat{\Sigma} \;:\; \dot{y} = f(y) - g(y). \tag{24}$$

For example, we consider the transition from "walk" motion to "squat" motion. The transient dynamics is described by

$$\begin{aligned}\dot{y} = {} & w_{\mathrm{ws}}(\tau)(f_{\mathrm{w}}(y) - g_{\mathrm{w}}(y)) \\ & + (1 - w_{\mathrm{ws}}(\tau))(f_{\mathrm{s}}(y) + g_{\mathrm{s}}(y)), \end{aligned} \tag{25}$$

where the subscription w (resp. s) means a function related to "walk" (resp. "squat"), and τ is set to be 0 when the transition of motion occurs and w_{ws} is the weight function:

$$w_{\mathrm{ws}}(\tau) = \frac{1}{\exp(\varepsilon_{\mathrm{ws}}\tau)}. \tag{26}$$

$\varepsilon_{\mathrm{ws}}$ represents the changing rate of vector fields. When the transition occurs, the trajectory starts to leave from the current motion and converge to the next motion, and after some time passed the vector fields of $\hat{\Sigma}_{\mathrm{w}}$ almost disappear and those of Σ_{s} are dominant. This transient dynamics enable trajectories to transfer between motions smoothly. Figures 8 (resp. Figure 9) shows a whole body motion from "walk" to "squat", where $\varepsilon_{\mathrm{ws}}$ equals 5 (resp. 500). In Figure 8, the changing rate of vector fields $\varepsilon_{\mathrm{ws}}$ is small, and a smooth transition occurs. On the other hand, in Figure 9, $\varepsilon_{\mathrm{ws}}$ is large, and a steep transition occurs. Thus, $\varepsilon_{\mathrm{ws}}$ can control the transient behavior between motions.

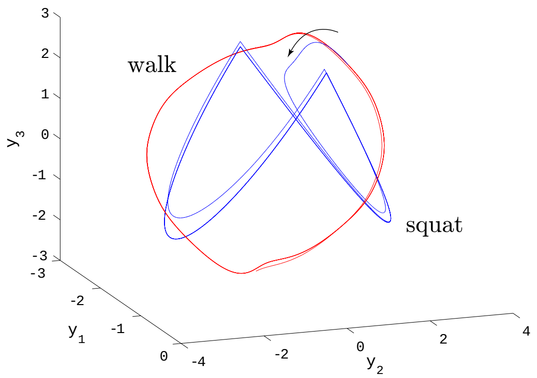

Figure 8: Transition of motion from "walk" to "squat" ($\varepsilon_{\mathrm{ws}} = 5$).

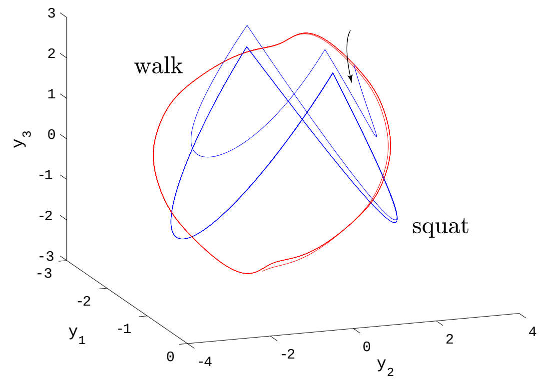

Figure 9: Transition of motion from "walk" to "squat" ($\varepsilon_{\mathrm{ws}} = 500$).

5. Conclusions

We have proposed a design method for a nonlinear system which generates a periodic motion of humanoid robots. First, for the smooth motion generation, we introduce a Lyapunov function based nonlinear system with a stable limit cycle based on conventional techniques. Second, for the nonsmooth motion generation, we employ piecewise quadratic Lyapunov function in order to realize nonsmoothness, and obtained a switched system. The proposed system, however, still have computational difficulties. Finally, the humanoid whole body motion is generated by combining designed systems. The transient behavior of motion is controlled by changing the stability of periodic motions gradually.

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