International Symposium on Nonlinear Theory and its Applications
Xi'an, PRC, October 7–11, 2002

Computation of Limit Cycles in a Class of Hybrid Dynamical Systems with Certain Symmetric Properties

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Abstract

This paper discusses the existence of limit cycles in a class of hybrid dynamical systems (HDSs), and derives a condition on system parameters for easily checking local stability. In this paper, we focus on piecewise affine (PWA) systems as a special class of HDSs. Assuming some symmetry on PWA systems, we obtain a simplified existence condition and a stability condition of limit cycles.

1. Introduction

Hybrid dynamical Systems (HDSs) are dynamical systems whose states consist of both continuous and discrete variables. While their discrete state is constant, their continuous state evolves according to a differential equation corresponding to the current discrete state. The discrete state may change to another one, when the continuous state satisfies a certain condition. Since many physical systems and engineering systems can be naturally represented as HDSs, analysis and design of HDSs are important.

Stability analysis for HDSs is generally very difficult since the conventional Lyapunov stability analysis is no more powerful. Recently, many approaches to stability analysis based on discontinuous Lyapunov functions or multiple ones have been developed [1, 2, 3, 4]. Moreover, it is a very difficult issue to compute limit cycles analytically. Limit cycles are one of the most important phenomena in nonlinear dynamical systems, and applied in many engineering fields. Limit cycles in continuous differential equations are smooth in general. But in applications such as generation of walking patterns for humanoid robots, nonsmooth limit cycles are needed. HDSs are useful for such limit cycles since change of the discrete states causes nonsmoothness.

In earlier studies, for example, the existence and stability of limit cycles in switched server system [5, 6], global asymptotical stability of limit cycles in relay feedback systems using extended Poincaré maps [7, 8] are reported. In more general case, modeling each point of trajectory on switching surface by discrete-time system, it is possible to check the exponential convergence of limit cycle by using discrete-time Lyapunov theory [9]. But in any case, it is still difficult to compute limit cycles, because it explicitly depends on the time. In this paper, we deal with hybrid dynamical systems whose continuous states are governed by piecewise affine (PWA) systems with certain symmetric properties, and propose an efficient method for computation of limit cycles. Furthermore, we derive a condition for the local stability of limit cycles which can be checked easily.

2. Piecewise Affine Systems

We consider the following PWA systems

$$\begin{aligned}\dot{x} &= A_{q(t)}x + B_{q(t)},\\ q(t) &\in Q \cup \mathtt{idle} \quad (Q := \{1, 2, \ldots, M\}),\end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the continuous state vector, $q(t) \in Q = \{1, 2, \cdots, M\}$ is the discrete state, $A_{q(t)} \in \mathbb{R}^{n\times n}$, and $B_{q(t)} \in \mathbb{R}^n$. In this paper, it is assumed that the discrete transition follows idling. Hence, during the idle time $\tau_{\mathtt{id}}$, the system is obeyed by $\dot{x} = A_{\mathtt{id}}x + B_{\mathtt{id}}$ ($\mathtt{id}$ represents $\mathtt{idle}$). The hybrid space is given by $\mathcal{H} = \mathbb{R}^n \times Q$. An initial state is assumed to be chosen in a set of possible initial state $(x_0, q_0) \in \mathcal{H}_0 \subset \mathcal{H}$. The continuous state $x(t)$ evolves according to $\dot{x} = A_{q_0}x + B_{q_0}$ from the given initial state. The discrete transition occurs when $x(\cdot)$ reaches a switch set S_{ij}, where S_{ij} represents the hypersurface on which change of the discrete state from i to j occurs. It is typically given by a hyperplane

$$S_{ij} = \{x \in \mathbb{R}^n \,|\, C_{ij}x = d_{ij}\}, \quad i, j \in Q. \tag{2}$$

Definition 1: *A solution $[x(t), q(t)]$ of (1) is said to be well-defined if the following conditions hold:*

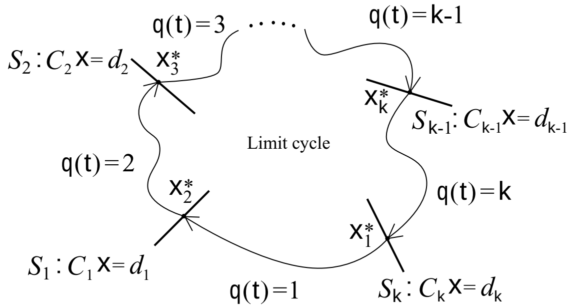

Figure 1: Cyclic change of discrete states.

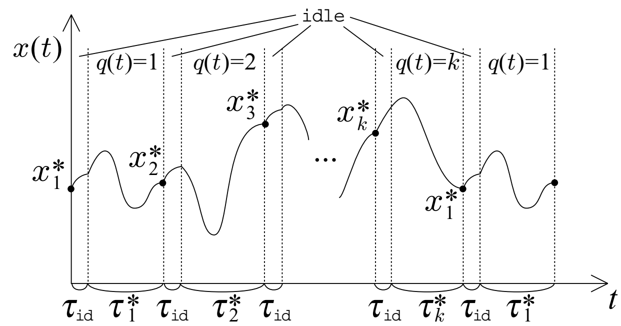

Figure 2: Trajectory of the limit cycle with idling.

(i) The solution is defined for all $t \in [0, \infty)$.

(ii) There exist a sequence $\{t_n\}_{n=0}^{\infty}$ such that $t_0 = 0$, $t_{n+1} > t_n$, $n = 0, 1, 2, \ldots$ and $\lim_{n\to\infty} t_n = \infty$.

$\{t_n\}_{n=0}^{\infty}$ is called the switching sequence of the solution $[x(t), q(t)]$.

Definition 2: *A well-defined solution $[x(t), q(t)]$ is said to be a periodic trajectory if there exists a time $T > 0$ such that $x(t+T) = x(t)$, $q(t+T) = q(t)$ for all $t \geq 0$*

Definition 3: *If a well-defined solution $[x(t), q(t)]$ is an isolated periodic trajectory, it said to be a limit cycle.*

Without loss of generality, if the system has a limit cycle, we can assume that the discrete state of limit cycle evolves $1 \to 2 \to \cdots k \to 1 \to 2 \cdots$ as shown in Fig. 1. Let x_i^* be the state of the limit cycle in the switch set S_i (we drop the subscript $i+1$ of $S_{i,i+1}$). Since the considered limit cycle has k discrete changes per one cycle, (1) is restricted as follows:

$$\begin{array}{c} \dot{x} = A_i x + B_i, \\ i \in \mathcal{I} \cup \texttt{idle} \quad (\mathcal{I} := \{1, 2, \ldots, k\}). \end{array} \tag{3}$$

It is noted that for a discrete state $i \in \mathcal{I}$, the continuous state trajectory of PWA system (3) which starts at $x_i(t_k)$ is given by

$$\begin{aligned} x_i(t) &= e^{A_i(t-t_k)} x_i(t_k) + \int_{t_k}^{t} e^{A_i(t-\tau)} B_i d\tau \\ &= e^{A_i(t-t_k)} \Big(x_i(t_k) + A_i^{-1} B_i \Big) - A_i^{-1} B_i, \\ & \qquad\qquad t \in [t_k, t_{k+1}]. \end{aligned} \tag{4}$$

3. Analysis of limit cycles

Assume that the PWA system (3) has a limit cycle, and this limit cycle crosses k switching surfaces per one cycle. To compute the limit cycle, by using (4), it is sufficient to solve the following set of equations:

$$\begin{cases} x_1^* = \mathcal{F}_{\texttt{id}} \circ \mathcal{F}_k \circ \mathcal{F}_{\texttt{id}} \circ \mathcal{F}_{k-1} \circ \cdots \circ \mathcal{F}_{\texttt{id}} \circ \mathcal{F}_1 \\ x_2^* = \mathcal{F}_{\texttt{id}} \circ \mathcal{F}_1 \circ \mathcal{F}_{\texttt{id}} \circ \mathcal{F}_k \circ \cdots \circ \mathcal{F}_{\texttt{id}} \circ \mathcal{F}_2 \\ \qquad \vdots \\ x_k^* = \mathcal{F}_{\texttt{id}} \circ \mathcal{F}_{k-1} \circ \mathcal{F}_{\texttt{id}} \circ \mathcal{F}_{k-2} \circ \cdots \circ \mathcal{F}_{\texttt{id}} \circ \mathcal{F}_k, \end{cases} \tag{5}$$

subject to $C_i x_{i+1}^* = d_i, i \in \mathcal{I}$ where mapping $\mathcal{F}_i$ is given, from (4), by

Case (i):$A_i \neq \mathbf{0}$, $i \in \mathcal{I} \cup \texttt{idle}$

$$\mathcal{F}_i(x_i^*) = e^{A_i \tau_i^*} \Big(x_i^* + A_i^{-1} B_i \Big) - A_i^{-1} B_i. \tag{6}$$

Case (ii):$A_i = \mathbf{0}$, $i \in \mathcal{I} \cup \texttt{idle}$

$$\mathcal{F}_i(x_i^*) = x_i^* + B_i \tau_i^*. \tag{7}$$

Here, τ_i^* denotes the duration where the discrete state i is active (Fig. 2).

Assumption 1: *The PWA system (3) satisfies the following symmetric property described, in terms of $J \in \mathbb{R}^{n\times n}$ and $r \in \mathbb{R}$ $(1 \leq r \leq k-1)$:*

$$\begin{array}{c} A_i = J^{k-r} A_{i-1} J^r, \quad B_i = J^{k-r} B_{i-1}, \\ C_{i,} = C_{i-1} J^r, \quad d_i = d, \quad i \in \mathcal{I}, \\ A_{\texttt{id}} = J^{k-r} A_{\texttt{id}} J^r, \quad B_{\texttt{id}} = J^{k-r} B_{\texttt{id}}. \end{array} \tag{8}$$

Let $\mathcal{C} : \mathbb{R}_+ \to \mathbb{R}^n$ be a periodic solution of system (3) with period T, and X^* be the closed orbit given by the image set $\mathcal{C}(t)$, that is,

$$X^* = \{x \in \mathbb{R}^n | x = \mathcal{C}(t), \, 0 \leq t \leq T\}. \tag{9}$$

Considering the idle time, the limit cycle starting at $x_1^* \in S_k \cap X^*$ satisfies $\mathcal{C}(\tau_{\texttt{id}} + \tau_1^*) = x_2^* \in S_1 \cap X^*$. Similarly, $\mathcal{C}(\tau_{\texttt{id}} + \tau_1^* + \tau_{\texttt{id}} + \tau_2^*) = x_3^* \in S_2 \cap X^*$, and so on. In the last discrete state k, the trajectory satisfies $\mathcal{C}(\sum_{i=1}^{k}(\tau_{\texttt{id}} + \tau_i^*)) = x_{k+1}^* = x_1^* \in S_k \cap X^*$. Note that $\sum_{i=1}^{k}(\tau_{\texttt{id}} + \tau_i^*) = T$.

Under Assumption 1, we have the following theorem which describes the relation between the state x_i^* and the duration τ_i^*.

Theorem 1: *PWA system (3) has a limit cycle X^* satisfying $x_{i+1}^* = J^{k-r}x_i^*$ if and only if all durations of each discrete state equal i.e., $\tau_0^* = \tau_1^* = \cdots = \tau_k^* = \tau^*$.*

Proof: See the Appendix.

By applying Theorem 1, the computation of limit cycles becomes easy as compared with solving (5). If $x_i^* \in S_{i-1} \cap X^*$, then x_{i+1}^* satisfies $x_{i+1}^* = J^{k-r}x_i^*$, that is,

$$J^{k-r}x_i^* = e^{A_i\tau^*}\Big(x_i^* + B_{\mathtt{id}}\tau_{\mathtt{id}} + A_i^{-1}B_i\Big) - A_i^{-1}B_i. \quad (10)$$

Here, we assume that a trajectory of the system is generated by $\dot{x} = B_{\mathtt{id}}$ when the discrete state is `idle`. Using (10), x_i^* is obtained by

$$x_i^* = \Big(J^{k-r} - e^{A_i\tau^*}\Big)^{-1}\Big(e^{A_i\tau^*} - I\Big)A_i^{-1}B_i + e^{A_i\tau^*}B_{\mathtt{id}}\tau_{\mathtt{id}}. \quad (11)$$

Furthermore, since x_i^* is in the switch set S_{i-1} and satisfies $C_{i-1}x_i^* = d_{i-1}$, τ^* is obtained by solving the equation

$$C_{i-1}\Big[\Big(J^{k-r} - e^{A_i\tau^*}\Big)^{-1}\Big(e^{A_i\tau^*} - I\Big)A_i^{-1}B_i + e^{A_i\tau^*}B_{\mathtt{id}}\tau_{\mathtt{id}}\Big] - d = 0. \quad (12)$$

It need not solve both (11) and (12) for each discrete state. If only one x_i^* is obtained, it is possible to compute other x_i^* recursively by using $x_{i+1}^* = J^{k-r}x_i^*$. From Theorem 1, the each duration τ_i^* $(i = 1, 2, \ldots, k)$ equals.

4. Stability of Limit Cycles

In this section, we present a simplified criterion for local stability of limit cycles. Limit cycles in HDSs are characterized by the state x_i^* belonging to the switch set S_{i-1}. it is possible to check the local stability of limit cycles by using the Poincaré map from a small neighborhood of x_i^* in S_{i-1} to the state when the trajectory returns to S_{i-1}. If all eigenvalues of the Jacobian matrix of the Poincaré map are inside the unit disk, the limit cycle is locally asymptotically stable.

Assume that the PWA system (3) has a limit cycle X^*. We consider a map from $x_i^* + \Delta_i x_i^*$ to $x_{i+1}^* + \Delta_{i+1}x_{i+1}^*$ after $\tau^* + \Delta_i t$ passed where Δ_i is chosen so that $x_i^* + \Delta_i x_i^*$ is in S_{i-1}. It is given (similar to [8]) by $\Delta_{i+1}x_{i+1}^* = W_i\Delta_i x_i^* + O(\Delta_0^2)$

$$W_i := \left\{I - \frac{\Big(A_i x_{i+1}^* + B_i\Big)C_i}{C_i\Big(A_i x_{i+1}^* + B_i\Big)}\right\}e^{A_i\tau^*}. \quad (13)$$

If all eigenvalues of $W = W_1W_2W_3\cdots W_k$ are inside the unit disk, the limit cycle $\mathcal{C}$ is locally asymptotically stable. Moreover in the PWA systems (3), W_i can be transformed as follows

$$\begin{aligned} W_i =& \left\{I - \frac{\Big(A_i x_{i+1}^* + B_i\Big)C_i}{C_i\Big(A_i x_{i+1}^* + B_i\Big)}\right\}e^{A_i\tau^*} \\ =& J^{k-r}\left\{I - \frac{\Big(A_{i-1}x_i^* + B_{i-1}\Big)C_{i-1}}{C_{i-1}\Big(A_{i-1}x_i^* + B_{i-1}\Big)}\right\}e^{A_{i-1}\tau^*}J^r \\ =& J^{k-r}W_{i-1}J^r. \end{aligned} \quad (14)$$

In the end, we get the Jacobian of the map W:

$$\begin{aligned} W =& W_kW_{k-1}\ldots W_2W_1 \\ =& J^{(k-r)(k-1)}W_1J^rk - 1\ \ldots\ J^{k-r}W_1J^r\ W_1 \\ =& J^{k(k-r-1)+r}W_1\big(J^rW_1\big)^{k-1} = \big(J^rW_1\big)^k. \end{aligned} \quad (15)$$

Thus, We have the following Theorem.

Theorem 2: *Consider the PWA systems (3). Assume there exists a limit cycle. If all eigenvalues of J^rW_1 are inside the unit disk, the limit cycle is locally asymptotically stable*

5. Illustrative Example

Consider the following piecewise affine system.

$$J = \begin{bmatrix} -0.5 & 0.866 \\ 0.866 & 0.5 \end{bmatrix}, \quad (J^2 = I),$$

$$A_1 = \begin{bmatrix} -1.5 & -5 \\ 3.5 & -2 \end{bmatrix}, B_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -1.225 & 3.659 \\ -4.842 & -2.275 \end{bmatrix}, B_2 = \begin{bmatrix} 1.366 \\ -0.366 \end{bmatrix},$$

$$A_{\mathtt{id}} = 0, B_{\mathtt{id}} = \begin{bmatrix} 10 \\ 17.320 \end{bmatrix},$$

$$S_1 = \Big\{x \in \mathbb{R}^2 \,\Big|\, \begin{bmatrix} -3 & -1 \end{bmatrix} x = 0\Big\},$$

$$S_2 = \Big\{x \in \mathbb{R}^2 \,\Big|\, \begin{bmatrix} 0.634 & -3.098 \end{bmatrix} x = 0\Big\}.$$

Let the idle time be $\tau_{\mathtt{id}} = 0.1$. Solving (11), (12) numerically and considering $x_{i+1}^* = J^{k-1}x_i^*$, we get

$$\tau^* = 0.263, \quad x_1^* = \begin{bmatrix} 2.247 \\ 0.460 \end{bmatrix}, \quad x_2^* = \begin{bmatrix} -0.725 \\ 2.176 \end{bmatrix}.$$

We can also check the local stability of this limit cycle by examining eigenvalues of JW_1.

$$JW_1 = \begin{bmatrix} 0.258 & 0.544 \\ 0.053 & 0.111 \end{bmatrix}, \ \lambda(JW_1) = 0,\ 0.369.$$

Since both of the eigenvalues are inside the unit disk, from Theorem 2 this limit cycle is locally asymptotically stable (Fig. 3).

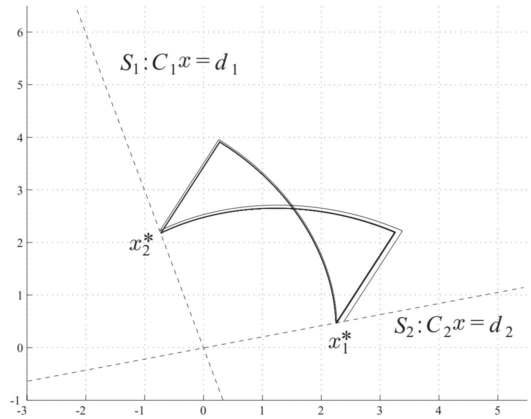

Figure 3: A simulated trajectory

6. Conclusions

Motivated by difficulties to compute limit cycles in HDSs, we have shown that the computation of the limit cycles are relaxed and to check the local stability becomes easy in PWA systems with certain symmetric properties. The results were applied to an example, but the constraints which system require is still strong.

Acknowledgments

This work has been supported by CREST of JST (Japan Science and Technology).

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Appendix: Proof of Theorem 1

In this proof, for simplicity, we consider only the case in this proof that the trajectory of system is given by $\dot{x} = B_{\mathtt{id}}$ (*i.e.*, $A_{\mathtt{id}} = \mathbf{0}$) during the idle time $\tau_{\mathtt{id}}$. Note that Theorem 1 can be proven similarly in case that $A_{\mathtt{id}} \neq \mathbf{0}$ by referring to (6).

Sufficiency: Assume all durations are same, that is, $\tau_0^* = \tau_1^* = \cdots = \tau_k^* = \tau^*$. Considering a state at time τ^* which starts at $x_i(0) \in S_{i-1,i}$, we get

$$\begin{aligned} x_i(\tau^*) &= e^{A_i\tau^*}\Big(x_i(0) + B_{\mathtt{id}}\tau_{\mathtt{id}} + A_i^{-1}B_i\Big) - A_i^{-1}B_i \\ &= J^r\Big\{e^{A_{i+1}\tau^*}\Big(J^{k-r}x_i(0) + B_{\mathtt{id}}\tau_{\mathtt{id}} + A_{i+1}^{-1}B_{i+1}\Big) \\ &\qquad - A_{i+1}^{-1}B_{i+1}\Big\}. \end{aligned}$$

Let $x(t)\big|_{x(0)}$ be a state at time t which starts from $x(0)$, that is

$$x_i(\tau^*)\big|_{x_i(0)} = J^r x_{i+1}(\tau^*)\big|_{x_{i+1}(0)=J^{k-r}x_i(0)}.$$

According to the sequence of discrete states, we get

$$\begin{aligned} x_i(\tau^*)\big|_{x_i(0)} &= J^r x_{i+1}(\tau^*)\big|_{x_{i+1}(0)=J^{k-r}x_i(0)} = \cdots \\ \cdots &= J^{kr}x_i(\tau^*)\big|_{x_i(0)=J^{k(k-r)}x_i(0)} = x_i(\tau^*)\big|_{x_i(0)}. \end{aligned}$$

Necessity: Assume that PWA system (3) with condition (8) has a limit cycle X^* satisfying $x_{i+1}^* = J^{k-r}x_i^*$. The state $x_{i+1}^* \in S_i$ reaches the next switch set S_{i+1} after τ_{i+1}^* passed, that is,

$$\begin{aligned} C_{i+1}\Big[e^{A_{i+1}\tau_{i+1}^*}\Big(x_{i+1}^* + B_{\mathtt{id}}\tau_{\mathtt{id}} + A_{i+1}^{-1}B_{i+1}\Big) \\ - A_{i+1}^{-1}B_{i+1}\Big] = d_{i+1} = d. \end{aligned}$$

Recall that $x_{i+1}^* = J^{k-r}x_i^*$. Thus,

$$\begin{aligned} C_i J^r\Big[J^{k-r}e^{A_i t}J^r\Big(J^{k-r}\Big(x_i^* + B_{\mathtt{id}}\tau_{\mathtt{id}}\Big) + J^{k-r}A_i^{-1}B_i\Big) \\ - J^{k-r}A_i^{-1}B_i\Big] = d, \\ \Leftrightarrow C_i\Big[e^{A_i t}\Big(x_i^* + B_{\mathtt{id}}\tau_{\mathtt{id}} + A_i^{-1}B_i\Big) - A_i^{-1}B_i\Big] = d = d_i. \end{aligned}$$

Finally, finding τ_{i+1}^* is equivalent to finding τ_i^*. □