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Synthesis of Hybrid Systems with Limit Cycles Using Piecewise Quadratic Lyapunov Functions

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1 Introduction

Limit cycles are one of the most important phenomena in nonlinear dynamical systems, and applied in many engineering fields. While stability analysis of limit cycles is a fundamental problem and many theories such as Lyapunov function methods have been proposed, the inverse problem of synthesizing a nonlinear system which has a stable and prescribed limit cycle is also important. Several methods for the inverse problem have been proposed [1, 2, 3, 4]. These methods are based on Lyapunov functions V and synthesized nonlinear systems have limit cycles satisfying $V(x) = c$ where x is a state variable and c is a given constant value.

On the other hand, Dynamical systems whose states consist of both continuous and discrete variables are called hybrid dynamical systems. Behaviors of their discrete state are piecewise-constant, and their continuous state evolves according to a differential equation corresponding to the current discrete state. Behaviors of the continuous state are inherently nonsmooth because of change of the discrete state. Many physical and mechanical systems can be naturally described by hybrid systems. In hybrid systems, stability analysis is very difficult since continuous Lyapunov functions are no more useful. Recently, many approaches to stability analysis based on discontinuous Lyapunov functions or multiple ones have been developed [5, 6, 7].

Since several hybrid systems do not have a constant steady state but a periodic one, studies of limit cycles in hybrid systems are more important. For example, the existence and stability of limit cycles in switched server system [8, 9], and global asymptotical stability of limit cycles in relay feedback systems using extended Poincaré maps [10] have been reported. In more general cases, discrete-time model is derived by focusing on points where behaviors hit switching surfaces, and it is possible to check the exponential convergence of limit cycle by using discrete-time Lyapunov functions [11]. However, there are little studies on how to construct a hybrid system which has a stable nonsmooth limit cycle. From a engineering viewpoint, such a limit cycle is applicable: for example, walking patterns of humanoid robots can be approximated by it [12, 13].

This paper proposes a synthesis method for hybrid systems with nonsmooth limit cycles. In the proposed method, a given periodic orbit is split into some ellipsoidal curves, we calculate a piecewise quadratic Lyapunov function $V(x)$ such that $V(x)$ is constant on the curve, and we obtain a desired hybrid system. The proposed method is an extension of Green's method [4].

This paper is organized as follows. In Section 2, we revisit and reformulate some useful techniques reported in [4] and show illustrative examples. In Section 3, hybrid systems derived from piecewise quadratic Lyapunov functions are presented and we discuss their properties. An example illustrates the results.

2 Systems with Prescribed Limit Cycles

In this section, we present several concepts that will be used throughout this paper. First, we consider the following continuous differential equations:

$$\dot{x} = f(x) + g(x), \quad f : \mathcal{R}^n \to \mathcal{R}^n, g : \mathcal{R}^n \to \mathcal{R}^n. \tag{1}$$

Here, we present sufficient conditions for the existence of an asymptotically stable limit cycle in (1).

Theorem 1 (Green [4]) *If there exists a continuously differentiable function $V : \Omega \to \mathcal{R}^m$ where Ω is a subset of $\mathcal{R}^n, (n > m)$ such that*

- $\dfrac{\partial V(x)}{\partial x} f(x) = 0, \quad \forall x \in \Omega.$
- *For each μth component of $V, 1 \le \mu \le m$,*

$$\frac{\partial V_\mu(x)}{\partial x} g(x) V_\mu(x) < 0, \quad \forall x \in \Omega \text{ such that } V_\mu(x(t)) \ne 0.$$

Then, (1) has an asymptotically stable limit cycle which satisfies $V(x) = 0$.

From Theorem 1, a trajectory of (1) starting from any initial point converges to the hypersurface $V(x) = 0$, and after the convergence the trajectory forms a closed curve on this hypersurface. As a special case, we consider that $m = 1$, and a system is described by the following affine form:

$$\dot{x} = Ax + a + V(x)(Bx + b), \tag{2}$$

where $x \in \mathcal{R}^n$, $A, B \in \mathcal{R}^{n\times n}$, $a, b \in \mathcal{R}^n$, and $V : \mathcal{R}^n \to \mathcal{R}$. In order to simplify the description of the system, we introduce an argumented state vector and rewrite (2) as follows:

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + V(x)\tilde{B}\tilde{x}, \tag{3}$$

where $\tilde{x} = \begin{bmatrix} x \\ 1 \end{bmatrix}, \tilde{A} = \begin{bmatrix} A & a \\ 0 & 0 \end{bmatrix}, \tilde{B} = \begin{bmatrix} B & b \\ 0 & 0 \end{bmatrix}$.

In (3), $V(x)$ is assumed to be

$$V(x) = x^T P x + 2p^T x + \pi = \tilde{x}^T \tilde{P} \tilde{x}, \tag{4}$$

where $P \in \mathcal{R}^{n\times n}$ is a positive definite symmetric matrix, $p \in \mathcal{R}^n$, $\pi \in \mathcal{R}$, and $\tilde{P} = \begin{bmatrix} P & p \\ p^T & \pi \end{bmatrix}$. Then, the following proposition is easily shown.

Proposition 1 *If there exists a symmetric matrix $\tilde{P}$ given by (4) such that*

- $\tilde{A}^T\tilde{P} + \tilde{P}\tilde{A} = 0.$
- $\tilde{B}^T\tilde{P} + \tilde{P}\tilde{B} < 0.$

Then, (3) has an asymptotically stable limit cycle.

For a given symmetric matrix $\tilde{P}$, we can construct a system with an asymptotically stable limit cycle by choosing matrices $\tilde{A}$ and $\tilde{B}$. Note that the matrix $\tilde{A}$ is simply given by

$$\tilde{A} = \tilde{G}_A\tilde{P}, \tag{5}$$

where $\tilde{G}_A = \begin{bmatrix} G_A & 0 \\ 0 & 0 \end{bmatrix}$ and G_A is an arbitrary skew-symmetric matrix. The matrix $\tilde{B}$ can be also chosen as follows:

$$\tilde{B} = \tilde{G}_B\tilde{P}, \tag{6}$$

where $\tilde{G}_B = \begin{bmatrix} G_B & 0 \\ 0 & 0 \end{bmatrix}$ and G_B is a matrix which satisfies $G_B^T + G_B < 0$.

Example 1 ($n = 2$) *Now, we consider the following symmetric matrix $\tilde{P}$.*

$$\tilde{P} = \begin{bmatrix} 7.5 & 1 & -2.7 \\ 1 & 2.5 & 0 \\ -2.7 & 0 & -5 \end{bmatrix}. \tag{7}$$

Then, we choose two matrices $\tilde{A}$ and $\tilde{B}$ such that these matrices satisfy (5) and (6). As an example, we set

$$\tilde{G}_A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{8}$$

$$\tilde{G}_B = \begin{bmatrix} -0.0462 & -0.002 & 0 \\ 0.0092 & -0.0314 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{9}$$

Then, we have

$$\tilde{A} = \begin{bmatrix} -1 & -2.5 & 0 \\ 7.5 & 1 & -2.7 \\ 0 & 0 & 0 \end{bmatrix}, \tag{10}$$

$$\tilde{B} = \begin{bmatrix} -0.3445 & -0.0412 & 0.1247 \\ 0.1004 & -0.0877 & 0.0248 \\ 0 & 0 & 0 \end{bmatrix}. \tag{11}$$

Example 2 ($n = 3$) *Set $\tilde{P}, \tilde{G}_A$, and $\tilde{G}_B$ as follows:*

$$\tilde{P} = \begin{bmatrix} 2 & 0.8 & 1.4 & -1.7 \\ 0.8 & 1 & 0.3 & 2 \\ 1.4 & 0.3 & 3 & -0.5 \\ -1.7 & 2 & -0.5 & -3 \end{bmatrix}, \tag{12}$$

$$\tilde{G}_A = \begin{bmatrix} 0 & 1 & 0.5 & 0 \\ -1 & 0 & 1 & 0 \\ -0.5 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{13}$$

$$\tilde{G}_B = \begin{bmatrix} -0.0185 & -0.0005 & 0.001 & 0 \\ 0.0015 & -0.0315 & -0.001 & 0 \\ -0.002 & -0.003 & -0.0005 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{14}$$

Then we have

$$\tilde{A} = \begin{bmatrix} 1.5 & 1.15 & 1.8 & 1.75 \\ -0.6 & -0.5 & 1.6 & 1.2 \\ -1.8 & -1.4 & -1 & -1.15 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{15}$$

$$\tilde{B} = \begin{bmatrix} -0.036 & -0.015 & -0.0231 & 0.03 \\ -0.0236 & -0.0306 & -0.0104 & -0.065 \\ -0.0071 & -0.0047 & -0.0052 & -0.0024 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{16}$$

Figures 1 and 2 show simulation results of Examples 1 and 2 from two initial states, respectively. Dashed lines in Figure 1 denote the level curves of the Lyapunov function. Both trajectories in this figure converge to the same limit cycle which satisfies $V(x) = 0$. In contrast, in Figure 2, two trajectories converge to different limit cycles which satisfy $V(x) = 0$ shown by an elliptic sphere.

It is clear from these results that this approach can not synthesize a prescribed limit cycle when $n > 2$, since the constraint $V(x) = 0$ defines an $(n-1)$-dimensional manifold and trajectories converge to limit cycles on the manifold depending on initial states. In order to determine one limit cycle in the

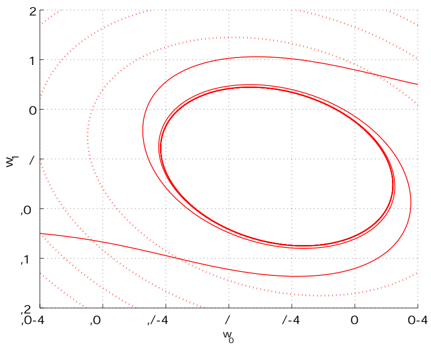

Figure 1: Trajectories of Example 1 from two initial conditions $x(0) = \pm[1.5 \quad 1.5]^T$.

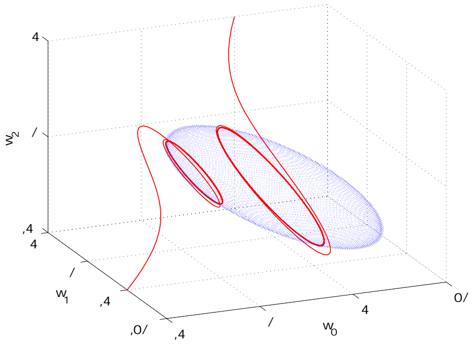

Figure 2: Trajectories of Example 2 from two initial conditions $x(0) = \pm[5 \quad 5 \quad 5]^T$.

case $n > 2$, (3) is modified as the following system with $V : \mathcal{R}^n \to \mathcal{R}^{n-1}$:

$$\dot{\tilde{x}} = \begin{bmatrix} A & \mathbf{0} & a \\ \Xi_1 & \mathbf{0} & \xi_1 \\ \vdots & \vdots & \vdots \\ \Xi_{n-2} & \mathbf{0} & \xi_{n-2} \\ [0 \;\; 0] & \mathbf{0} & 0 \end{bmatrix} \tilde{x} + \begin{bmatrix} V_1 \left[B \;\; \mathbf{0} \;\; b\right] \tilde{x} \\ \alpha_1 V_2 \\ \vdots \\ \alpha_{n-2} V_{n-1} \\ 0 \end{bmatrix}, \quad (17)$$

where $\tilde{x} = [x_1 \, x_2 \cdots x_n \; 1]^T$, $A, B \in \mathcal{R}^{2\times 2}$, $a, b \in \mathcal{R}^{2\times 1}$, $\Xi_i \in \mathcal{R}^{1\times 2}$, $\xi_i, \alpha_i \in \mathcal{R}, i = 1, \ldots, n-2$, and $V = [V_1 \; V_2 \; \cdots V_{n-1}]^T : \mathcal{R}^n \to \mathcal{R}^{n-1}$. $V_i(x)$ is given by

$$V_i(x) = \begin{cases} \begin{bmatrix} x_1 & x_2 & 1 \end{bmatrix} \begin{bmatrix} P & p \\ p^T & \pi \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix}, & \text{if } i = 1, \\ \zeta_i x_1 + \eta_i x_2 + \lambda_i - x_{i+1}, & \text{otherwise.} \end{cases} \quad (18)$$

$V_1(x)$ defines an elliptic cylinder and the other functions $V_i(x)$ define hyperplanes in the n-dimensional

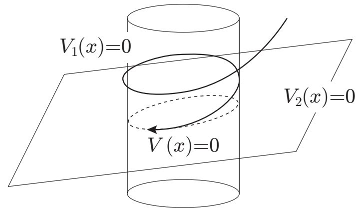

Figure 3: The constraint $V(x) = 0$ in 3-dimensional space.

space. Figure 3 shows the relation between $V_1(x)$ and $V_2(x)$. To construct a system (17) with an asymptotically stable limit cycle which satisfies $V(x) = 0$ (this defines a 1-dimensional manifold), we consider the condition $\frac{\partial V(x)}{\partial x} f(x) = 0$. The matrix A and the vector a can be determined by (5). For the other parameters $\Xi_i, \xi_i, i = 1 \ldots n-2$,

$$\frac{\partial V_i(x)}{\partial x} f(x) = \begin{bmatrix} \zeta_i & \eta_i & -1 \end{bmatrix} \begin{bmatrix} A & a \\ \Xi_{i-1} & \xi_{i-1} \end{bmatrix} = 0, \quad (19)$$

which implies

$$\begin{bmatrix} \Xi_{i-1} & \xi_{i-1} \end{bmatrix} = \begin{bmatrix} \zeta_i & \eta_i \end{bmatrix} \begin{bmatrix} A & a \end{bmatrix}. \quad (20)$$

Thus, using (5) and (20), we can construct (17) which has a limit cycle specified by the intersection of V_i. Next we consider the second condition $\frac{\partial V_i(x)}{\partial x} g(x) V_i(x) < 0$. Here, we introduce the following proposition [4, Corollary 2.1].

Proposition 2 *Assume that Theorem 1 is applicable to V_1. If all trajectories of (1) are bounded, the second condition of Theorem 1 is modified as follows:*

- *For each μth component of $V_\mu, 2 \le \mu \le n-1$,*

$$\frac{\partial V_\mu(x)}{\partial x} g(x) V_\mu(x) < 0, \quad \forall x \in \Omega \text{ such that } V_1(x) = 0.$$

By choosing the matrix B and the vector b from (6), it is guaranteed that all trajectories converge to $V_1(x) = 0$ as $t \to \infty$. When $V_1(x) = 0$,

$$\frac{\partial V_i(x)}{\partial x} g(x) V_i(x) = \begin{bmatrix} \zeta_i & \eta_i & -1 \end{bmatrix} \begin{bmatrix} V_1 \left[B \;\; \mathbf{0} \;\; b\right] \tilde{x} \\ \alpha_{i-1} V_i \end{bmatrix} \alpha_{i-1} V_i(x) = -\alpha_{i-1} V_i(x)^2 < 0. \quad (21)$$

Hence, once B and b are determined, the convergence condition $\frac{\partial V_i(x)}{\partial x} g(x) V_i(x) < 0$ is satisfied automatically in (17) from Proposition 2, and trajectories converge to one limit cycle. It is noted that $\alpha_{i-1} > 0$ represents a convergence rate of V_i.

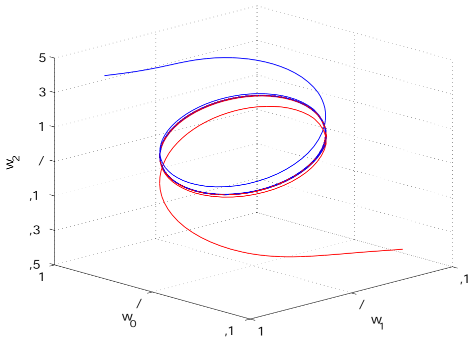

Figure 4: Trajectories of Example 3 from two initial conditions $x(0) = \pm[1.5 \;\; 1.5 \; 5]^T$.

Example 3 *We consider the case when* $V : \mathcal{R}^3 \to \mathcal{R}^2$. *Let* $V_1(x)$ *be (7) and*

$$V_2(x) = 1.5x_1 - 0.2x_2 - x_3. \tag{22}$$

Set matrices A, B *and vectors* a, b *as (10) and (11). Using (20), we have*

$$\begin{bmatrix}\Xi_1 & \xi_1\end{bmatrix} = \begin{bmatrix}3 & 3.95 & -0.54\end{bmatrix}. \tag{23}$$

Figure 4 shows a simulation result of this example. These trajectories converge to the same limit cycle.

3 Synthesis Method

We consider hybrid systems described by

$$\begin{cases} \dot{x}(t) = f(x(t), q(t)), \\ q^+(t) = \phi(x(t), q(t)), \end{cases} \tag{24}$$

where $x \in \mathcal{R}^n$ is the continuous state vector, $q \in Q = \{1, 2, \cdots, M\}$ is the discrete state, $q^+(t)$ refers to the lefthand limit of the function $q(t)$ at time t, that is $q^+(t) = \lim_{\varepsilon \to +0} q(t + \varepsilon)$. The function $\phi : \mathcal{R}^n \times Q \to Q$ describes the change of the discrete state, and a switching of discrete state from q to r is described by a switch set $S_{q,r}$:

$$S_{q,r} = \{x \in \mathcal{R}^n \,|\, \phi(x, q) = r\}, \quad q, r \in Q. \tag{25}$$

The function $f : \mathcal{R}^n \times Q \to \mathcal{R}^n$ shows a vector field, and the continuous state $x(\cdot)$ evolves according to $f(\cdot, q)$ for each state $q \in Q$. In this paper each $f(\cdot, q)$ is called a subsystem q.

Definition 1 *A solution* $(x(t), q(t))$ *of (24) is said to be well-defined if the following conditions hold:*

(i) The solution is defined for $t \in [0, \infty)$.

(ii) There exist a sequence $\{t_n\}_{n=0}^{\infty}$ *such that* $t_0 = 0$, $t_{n+1} > t_n$, $n = 0, 1, 2, \ldots$, $\lim_{n\to\infty} t_n = \infty$, *and* $q(t)$ *is discontinuous at* $t_n, n = 1, 2, \ldots$.

$\{t_n\}_{n=0}^{\infty}$ *is called the switching sequence of the solution* $(x(t), q(t))$.

Definition 2 *A well-defined solution* $(x(t), q(t))$ *is said to be a periodic trajectory if there exists a time* $T > 0$ *such that* $x(t + T) = x(t)$, $q(t + T) = q(t)$ *for all* $t \geq 0$

Definition 3 *If a well-defined solution* $(x(t), q(t))$ *is an isolated periodic trajectory, it said to be a limit cycle.*

The hybrid space of (24) is given by $\mathcal{H} := \mathcal{R}^n \times Q$. Consider an initial state which lies in a set of possible initial conditions $(x_0, q_0) \in \mathcal{H}_0 \subset \mathcal{H}$, and assume that a trajectory $(x(t), q(t))$ starting from (x_0, q_0) is well-defined. The trajectory of (24) evolves according to $\dot{x} = f(x, q_0)$, and if a state $x(t)$ hits a switch set $S_{q,r}$ at time t_n, the corresponding discrete transition from the discrete state q to r occurs. The evolution of the discrete state can be described by a sequence as follows:

$$\xi(x_0, q_0) = (q_0, t_0), (q_1, t_1), \ldots, \tag{26}$$

where (q_k, t_k) means that $\dot{x} = f(x(t), q_k)$ for $t_k \leq t < t_{k+1}$ and $q^+(t_k) = g(x(t), q_k) = q_{k+1}$. For (26), we define the following projection to a time sequence:

$$\xi_t(x_0, q_0) = t_0, t_1, t_2 \ldots. \tag{27}$$

To express a sequence of the time interval where discrete state equals q, we define the following projection:

$$\xi_t(x_0, q_0)|q = t_0^q, t_1^q, \ldots, t_{2k}^q, t_{2k+1}^q, \ldots, \quad k \in N, \tag{28}$$

where t_{2k}^q and t_{2k+1}^k are time instances where the subsystem q is switched on and off, respectively. Furthermore, to obtain the duration which the system is driven by the subsystem q, we define the interval completion $I(\xi_t(x_0, q_0)|q)$ as a set obtained by taking the union of all close intervals

$$I(\xi_t(x_0, q_0)|q) = \bigcup_{k\in N} [t_{2k}^q, t_{2k+1}^q]. \tag{29}$$

Denote $E(\xi_t(x_0, q_0)|q)$ as the even sequence of $\xi_t(x_0, q_0)|q$

$$E(\xi_t(x_0, q_0)|q) = t_0^q, t_2^q, \ldots, t_{2k}^q, \ldots, \quad k \in N. \tag{30}$$

The conditions of Theorem 1 are based on only one Lyapunov function. We will show that Theorem 1 is extended to the form for hybrid systems. We consider the following hybrid system;

$$\begin{cases} \dot{x}(t) = f(x(t), q(t)) + g(x(t), q(t)), \\ q^+(t) = \phi(x(t), q(t)). \end{cases} \tag{31}$$

Theorem 2 *For all hybrid sequences* $\xi(x_0, q_0)$ *of (31) which start from any initial consition* $(x_0, q_0) \in \mathcal{H}_0$*, if there exists a continuously differentiable function* $\mathcal{V}_q : \Omega_q \to \mathcal{R}^m$*, for all* $q \in Q$ *where* Ω_q *is a subset of* $\mathcal{R}^n, (n > m)$ *such that*

- $$\frac{\partial \mathcal{V}_q(x)}{\partial x} f(x,q) = 0, \quad \forall x \in \Omega_q, \forall t \in I(\xi_t(x_0,q_0)|q).$$
- *For each* μ*th componet of* $\mathcal{V}_q, 1 \le \mu \le m$,
$$\frac{\partial \mathcal{V}_{q_\mu}(x)}{\partial x} g(x,q)\mathcal{V}_{q_\mu}(x) < 0, \quad \forall x \in \Omega_q \ s.t.\ \mathcal{V}_{q_\mu}(x) \neq 0,\ \forall t \in I(\xi_t(x_0,q_0)|q).$$
- $\mathcal{V}_q(x) = \mathcal{V}_r(x), \quad x \in S_{q,r},\ \forall t \in E(\xi_t(x_0,q_0)|q).$

Then, (31) has an asymptotically stable limit cycle.

For hybrid systems, it is common to use multiple or piecewise quadratic Lyapunov functions [6, 7]. We consider the following hybrid system with $\mathcal{V}_{q_i}$ given by (18).

$$\begin{cases} \dot{\tilde{x}} = \begin{bmatrix} A_q & \mathbf{0} & a_q \\ \Xi_{q_1} & \mathbf{0} & \xi_{q_1} \\ \vdots & \vdots & \vdots \\ \Xi_{q_{n-2}} & \mathbf{0} & \xi_{q_{n-2}} \\ [0\ \ 0] & \mathbf{0} & 0 \end{bmatrix} \tilde{x} + \begin{bmatrix} \mathcal{V}_{q_1}\begin{bmatrix} B & \mathbf{0} & b \end{bmatrix}\tilde{x} \\ \alpha_{q_1}\mathcal{V}_{q_2} \\ \vdots \\ \alpha_{q_{n-2}}\mathcal{V}_{q_{n-1}} \\ 0 \end{bmatrix}, \\ q^+(t) = r, \quad \text{if} \ \ q(t) = q \ \ \text{and} \ \ x(t) \in S_{q,r}, \end{cases} \tag{32}$$

where the transition of discrete state from q to r occurs whenever trajectories hit a hyperplane

$$S_{q,r} = \{x \in \mathcal{R}^n | \tilde{c}_{q,r}^T \tilde{x} = 0\}, \quad q, r \in Q, \tag{33}$$

where $\tilde{c} = \begin{bmatrix} c_{q,r}^T & d_{q,r} \end{bmatrix}^T$, $c_{q,r} \in \mathcal{R}^n$, and $d_{q,r} \in \mathcal{R}$. For given $\mathcal{V}_q$, this hybrid system satisfies both the first and second condition in Theorem 2 by determining each parameter from (5), (6), (20) and (21). But $\mathcal{V}_q$ is not allowed to choice freely because of the third condition. This condition requires the continuity of Lyapunov functions on all switching surfaces, and it is hard to find such Lyapunov functions in general. From Proposition 2, however, if $\mathcal{V}_{q_1}$ satisfies the continuity on all switching surfaces, the second and third condition of Theorem 2 is relaxed.

Since $\mathcal{V}_{q_1}$ is quadratic, this type of Lyapunov functions can be formulated by using the conditions for discrete transition [6]. Let the Lyapunov function be $\mathcal{V}_{q_1}(x) = \tilde{x}^T \tilde{P}_q \tilde{x}$ and consider the case that the discrete state switches from q to r. The condition of switching is given by $\tilde{c}_{q,r}^T \tilde{x} = 0$. Since Lyapunov functions should be continuous on switching surfaces, the third condition in Theorem 2 can be written as

$$\tilde{P}_r = \tilde{P}_q + \tilde{t}_{q,r}^T \tilde{c}_{q,r} + \tilde{c}_{q,r}^T \tilde{t}_{q,r}, \tag{34}$$

where $\tilde{t}_{i,j}$ is an $(n+1)$-dimensional vector.

In a piecewise affine system which is a special class of hybrid systems, the state space is partitioned into several regions $X_q \subseteq \mathcal{R}^n, q \in Q$, and each discrete state is defined by the region. If each region forms a polyhedron with pairwise disjoint interior, we can obtain the following matrices for each region:

$$\tilde{E}_q = \begin{bmatrix} E_q & e_q \end{bmatrix}, \quad \tilde{F}_q = \begin{bmatrix} F_q & f_q \end{bmatrix},$$

where

$$\tilde{E}_q \tilde{x} \ge 0, \quad x \in X_q, \tag{35}$$

$$\tilde{F}_q \tilde{x} = \tilde{F}_r \tilde{x}, \quad x \in X_q \cap X_r. \tag{36}$$

In this paper, these polyhedrons are given by

$$\tilde{E}_q = \begin{bmatrix} E_{q_1} & E_{q_2} & 0 & \cdots & 0 & e_q \end{bmatrix}, \tag{37}$$

where E_{q_i} is the ith column vector of E_q, since we consider a two-dimensional quadratic Lyapunov function independently of the dimension of the system

By using this representation, the requirement that a Lyapunov function is continuous at every point on the switching surface can be written as

$$\tilde{P}_q = \tilde{F}_q T \tilde{F}_q, \tag{38}$$

where T is a symmetric matrix. Once the continuity with respect to $\mathcal{V}_{q_1}$ is assured by using (34) or (38), from Proposition 2 we have the following proposition.

Proposition 3 *Assume* $\mathcal{V}_{q_1}$*, for all* $q \in Q$*, satisfies all conditions of Theorem 2. If all trajectories of (32) are well-defined, the second and third conditions of Theorem 2 respect to* $\mathcal{V}_{q_\mu}$*,* $2 \le \mu \le n-1$*, is simplified as follows:*

- *For each* μ*th componet of* $\mathcal{V}_{q_\mu}, 2 \le \mu \le n-1$,
$$\frac{\partial \mathcal{V}_{q_\mu}(x)}{\partial x} g(x,q)\mathcal{V}_{q_\mu}(x) < 0, \quad \forall x \in \Omega_q \ s.t.\ \mathcal{V}_{q_1}(x) = 0,\ \forall t \in I(\xi_t(x_0,q_0)|q).$$
- *For each* μ*th componet of* $\mathcal{V}_{q_\mu}, 2 \le \mu \le n-1$,
$$\mathcal{V}_{q_\mu}(x) = \mathcal{V}_{r_\mu}(x),\ x \in S_{q,r} \ s.t.\ \mathcal{V}_{q_1}(x) = \mathcal{V}_{r_1}(x), \quad \forall t \in E(\xi_t(x_0,q_0)|q).$$

Example 4 *We synthesize a hybrid system whose trajectories converge to the following limit cycle.*

$$\mathcal{V}_1(x) = \begin{bmatrix} 6x_1^2 + 2x_2^2 - 8 \\ x_2 - x_3 \end{bmatrix}, \quad \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \end{bmatrix} x \ge 0,$$

$$\mathcal{V}_2(x) = \begin{bmatrix} 2x_1^2 + 6x_2^2 - 8 \\ x_1 - x_3 \end{bmatrix}, \quad \begin{bmatrix} 1 & -1 & 0 \\ -1 & -1 & 0 \end{bmatrix} x \ge 0,$$

$$\mathcal{V}_3(x) = \begin{bmatrix} 6x_1^2 + 2x_2^2 - 8 \\ -x_2 - x_3 \end{bmatrix}, \quad \begin{bmatrix} -1 & 1 & 0 \\ -1 & -1 & 0 \end{bmatrix} x \ge 0,$$

$$\mathcal{V}_4(x) = \begin{bmatrix} 2x_1^2 + 6x_2^2 - 8 \\ -x_1 - x_3 \end{bmatrix}, \quad \begin{bmatrix} -1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} x \ge 0,$$

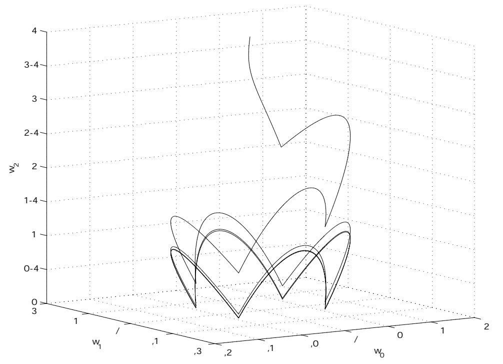

Figure 5: Trajectory of Example 4 from initial condition $x(0) = [0\ 0.5\ 5]^T$.

In Example 4, since $\mathcal{V}_{q_1}$, $q \in \{1, 2, 3, 4\}$, are continuous on all switching surfaces, and $\mathcal{V}_{q_2}$ are also continuous under the condition $\mathcal{V}_{q_1} = \mathcal{V}_{r_1}$, we can construct a hybrid system with an asymptotically stable limit cycle. A trajectory converges to the desired limit cycle as shown in Figure 5.

4 Conclusions

In this paper, we propose a synthesis of hybrid systems with limit cycles defined by a constraint $V(x) = 0$. Limit cycles of designed hybrid systems are composed by elliptic cylinders and hyperplanes. Synthesis of hybrid systems with more general form of limit cycles is a future study.

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